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AN EXACT SOLUTION OF 4D HIGHER-SPIN GAUGE THEORY

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ABSTRACT

We give a one-parameter family of exact solutions to 4D higher-spin gauge theory invariant under a deformed higher-spin extension of $SO(3, 1)$ and parameterized by a zero-form invariant. All higher-spin gauge fields vanish, while the metric interpolates between two asymptotically AdS_4 regions via two dS_3 -foliated domain walls and two H_3 -foliated Robertson-Walker spacetimes – one in the future and one in the past – with the scalar field playing the role of foliation parameter. All Weyl tensors vanish, including that of spin two. We furthermore discuss methods for constructing solutions, including deformation of solutions to pure AdS gravity, the gauge-function approach, the perturbative treatment of (pseudo-)singular initial data describing isometric or otherwise projected solutions, and zero-form invariants.

Contents

1	INTRODUCTION AND SUMMARY	3
2	THE MINIMAL BOSONIC MODEL	5
2.1	THE MASTER-FIELD EQUATIONS	5
2.2	THE SPACE-TIME FIELD EQUATIONS	8
3	CONSTRUCTION OF SOLUTIONS	10
3.1	PERTURBATIVE SPACE-TIME APPROACH	10
3.2	THE Z -SPACE APPROACH	12
3.3	ON REGULAR, SINGULAR AND PSEUDO-SINGULAR INITIAL CONDITIONS	14
3.4	ZERO-FORM CURVATURE INVARIANTS	16
4	AN $SO(3, 1)$-INVARIANT SOLUTION	18
4.1	THE ANSATZ	18
4.2	SOLUTION OF Z -SPACE EQUATIONS	20
4.3	THE SOLUTION IN SPACETIME	24
4.4	SYMMETRIES OF THE SOLUTION	27
5	ON OTHER SOLUTIONS WITH NON-MAXIMAL ISOMETRY	28
5.1	SOME GENERALITIES	28
5.2	SOLUTIONS WITH UNBROKEN $SO(3, 1)$ SYMMETRY	30
5.3	ON DOMAIN WALLS, ROTATIONALLY INVARIANT AND RW-LIKE SOLUTIONS	32
5.4	EXISTENCE OF SYMMETRY PARAMETERS TO THE SECOND ORDER	36
6	DISCUSSION	38
A	CONVENTIONS AND NOTATION	40

B GAUSSIAN INTEGRATION FORMULAE	41
C SECOND-ORDER CORRECTIONS TO THE FIELD EQUATIONS	41
D ANALYSIS OF THE Q-FUNCTION	43

1 INTRODUCTION AND SUMMARY

Full higher-spin gauge-field equations have been known in $D = 4$ for quite some time, essentially since the early work of Vasiliev [1] (see [2] for a review), and their generalizations to higher dimensions have been started to be understood more recently in [3, 4, 5]. These equations are generalizations of pure AdS gravity, in which the metric is accompanied by an infinite tower of higher-spin fields *and* special sets of lower-spin fields, always containing at least one real scalar. In the minimal setting, the physical fields are symmetric doubly traceless Lorentz tensors $\phi_{\mu_1 \dots \mu_s}$ of rank $s = 0, 2, 4, \dots$. The generally covariant form of the equations [6] (with non-linearly treated metric $ds^2 = dx^\mu dx^\nu g_{\mu\nu}$ accommodating the rank-two tensor) is highly non-local with higher-derivative corrections normalized by the AdS mass-parameter [1, 2]. The expansion around the AdS vacuum yields, however, a tachyon and ghost-free spectrum of massless particles governed by Fronsdal’s equations, with all non-localities occurring via interactions. In other words, the classical higher-spin gauge theory is weakly coupled in field amplitudes and strongly coupled in wave-numbers. This state of affairs – which refers to a classical theory although it appears reminiscent to that of a quantum-effective theory – blurs many basic field-theoretic concepts, even at the level of the lower-spin self-couplings, which, for example, no longer exhibit any well-defined notion of microscopic scalar-field potential [7] or local stress-energy tensor [8]. The non-localities are, of course, not put in by hand, nor by integrating out microscopic fields. Instead, the complete form of the interactions are governed – somewhat in the spirit of classical string field theory – by a manifestly higher-spin covariant master-field formulation [1, 2] (without ambiguities in case manifest Lorentz invariance [2] and parity invariance [9, 7] is required) based on simple non-commutative twistor variables [1] (see also [10, 11, 12, 13, 14, 15] for higher-dimensional generalizations albeit at the free level) related to deformation quantization of singletons [16, 17].

The master-field formulation, which is presented in Section 2, is amenable to finding classical solutions, while the non-localities make these difficult to interpret using traditional field-theoretic and geometric tools. Clearly, the finding of exact solutions is therefore highly desirable, not only for the important role that they may play in uncovering novel physical phenomena, but also for developing the interpretational

language as well as shedding new light on the origin of the higher-spin field equations themselves in tensionless limits of String Theory [18, 19, 20, 21, 22, 17].

It is primarily with the above motivations in mind that we have sought and found an exact solution beyond AdS spacetime. In doing so, we have greatly benefitted from the work of [23, 24, 2] on how to use the master-field formulation to construct solutions in ordinary spacetime. Exact, massively deformed, 3D vacua are constructed in [23] using techniques for deforming twistor oscillators, and plane-wave solutions for the free 4D field equations are provided in [24] using a gauge-function approach, in its turn related to a more general method described in [23, 2] pertinent to finding exact solutions in a systematic manner by means of integrating flows. We shall relate further to these works when we discuss methods for finding solutions in Section 3.

The solution, which is presented in detail in Section 4, is constructed by exploiting the simplifications taking place at the full master-field level by imposing $SO(3, 1)$ -invariance (without having to resort to weak-field expansion). As a result, it is globally well-defined on a manifold with the same topology as AdS_4 , covered by two charts with local stereographic coordinates x^a and \tilde{x}^a (restricted by $\lambda^2 x^2, \lambda^2 \tilde{x}^2 < 1$ and related by $\tilde{x}^a = -x^a/(\lambda^2 x^2)$ in the overlap region $\lambda^2 x^2, \lambda^2 \tilde{x}^2 < 0$). In the Type A model (see Section 2.1), the solution is given locally in the x^a -chart by

$$\phi(x) = \frac{\nu}{b_1}(1 - \lambda^2 x^2), \quad ds^2 = \frac{4\Omega^2(d(g_1 x))^2}{(1 - \lambda^2 g_1^2 x^2)^2}, \quad \omega_\mu^{ab} = f\omega_{(0)\mu}^{ab}, \quad (1.1)$$

$$\phi_{\mu_1 \dots \mu_s} = 0, \quad \text{for } s = 4, 6, \dots, \quad (1.2)$$

where the scale factors $\Omega(x^2; \nu)$, $g_1(x^2; \nu)$ and $f(x^2; \nu)$ are given explicitly in (4.63) and (4.66), and λ is the inverse radius of the AdS_4 vacuum at $\nu = 0$, where $\Omega(x^2; 0) = f(x^2; 0) = g_1(x^2; 0) = 1$.

Remarkably, the higher-spin fields vanish, notwithstanding the fact that lower spins in general source higher spins, so that the field equations can only be truncated to the spin $s \leq 2$ sector for very special configurations. In other words, the solution satisfies a highly non-local extension of scalar-coupled AdS gravity (corresponding to an effective model given in [25]) as well as an infinite set of consistency conditions for setting the higher-spin fields to zero. These conditions reflect the invariance under an infinite-dimensional higher-spin extension of $SO(3, 1)$, that we denote by

$$hsl(2, C; \nu) \supset sl(2, C), \quad (1.3)$$

with ν -dependent generators given in (4.71) and (4.72). This motivates seeking solutions based on invariance under other subgroups of $SO(3, 2)$, which we shall analyze at the linearized level in Section 5.

The locally defined scalar and metric in (1.1) are components of a section of the higher-spin gauge-covariant master fields, related to the section in the \tilde{x}^a -chart by a gauge transition defined in the overlap. The physical component fields are thus related by more complicated “duality” transformations than standard reparameterizations. As a result, by the Z_2 -symmetry, they read the *same* in x^a and \tilde{x}^a -coordinates, and hence the scalar-field duality transformation (in this particular background) assumes the form of a fractional linear transformation, *viz.*

$$\tilde{\phi}(\tilde{x}) = \frac{\nu\phi(x)}{\phi(x) - \nu}. \quad (1.4)$$

The global solution therefore remains “weakly coupled” throughout spacetime, and describes a Weyl-flat interpolation between two asymptotically AdS_4 regions $\lambda^2 x^2 \sim 1$ and $\lambda^2 \tilde{x}^2 \sim 1$, via two dS_3 -foliated domain walls in $0 < \lambda^2 x^2, \lambda^2 \tilde{x}^2 < 1$ and two FRW spacetimes with $k = -1$ in the overlap region (one in the future and one in the past).

As we shall discuss in Section 6, the cosmological interpretation of our solution requires the development of geometric and algebraic tools suitable for the description of higher gauge theory. Some of the properties of the solution aiming at such an interpretation are reported in [25].

2 THE MINIMAL BOSONIC MODEL

2.1 THE MASTER-FIELD EQUATIONS

To describe the higher-spin gauge theory based on the minimal higher-spin algebra $hs(4) \supset SO(3, 2)$, one introduces a set of auxiliary coordinates $(z^\alpha, \bar{z}^{\dot{\alpha}})$ together with an additional set of internal variables $(y_\alpha, \bar{y}_\alpha)$, that are Grassmann-even $SL(2, C)$ -spinor oscillators defined by the associative product rules

$$y_\alpha \star y_\beta = y_\alpha y_\beta + i\epsilon_{\alpha\beta}, \quad y_\alpha \star z_\beta = y_\alpha z_\beta - i\epsilon_{\alpha\beta}, \quad (2.1)$$

$$z_\alpha \star y_\beta = z_\alpha y_\beta + i\epsilon_{\alpha\beta}, \quad z_\alpha \star z_\beta = z_\alpha z_\beta - i\epsilon_{\alpha\beta}, \quad (2.2)$$

where the juxtaposition denotes the symmetrized, *i.e.* Weyl-ordered, products. The hermitian conjugates $(y_\alpha)^\dagger = \bar{y}_{\dot{\alpha}}$ and $(z_\alpha)^\dagger = \bar{z}_{\dot{\alpha}}$ obey

$$\bar{y}_{\dot{\alpha}} \star \bar{y}_{\dot{\beta}} = \bar{y}_{\dot{\alpha}} \bar{y}_{\dot{\beta}} + i\epsilon_{\dot{\alpha}\dot{\beta}}, \quad \bar{z}_{\dot{\alpha}} \star \bar{y}_{\dot{\beta}} = \bar{z}_{\dot{\alpha}} \bar{y}_{\dot{\beta}} - i\epsilon_{\dot{\alpha}\dot{\beta}}, \quad (2.3)$$

$$\bar{y}_{\dot{\alpha}} \star \bar{z}_{\dot{\beta}} = \bar{y}_{\dot{\alpha}} \bar{z}_{\dot{\beta}} + i\epsilon_{\dot{\alpha}\dot{\beta}}, \quad \bar{z}_{\dot{\alpha}} \star \bar{z}_{\dot{\beta}} = \bar{z}_{\dot{\alpha}} \bar{z}_{\dot{\beta}} - i\epsilon_{\dot{\alpha}\dot{\beta}}. \quad (2.4)$$

Equivalently, in terms of Weyl-ordered functions

$$\hat{f}(y, \bar{y}, z, \bar{z}) \star \hat{g}(y, \bar{y}, z, \bar{z}) \quad (2.5)$$

$$= \int \frac{d^2\xi d^2\eta d^2\bar{\xi} d^2\bar{\eta}}{(2\pi)^4} e^{i\eta^\alpha \xi_\alpha + i\bar{\eta}^\dot{\alpha} \bar{\xi}_\dot{\alpha}} \hat{f}(y + \xi, \bar{y} + \bar{\xi}, z + \xi, \bar{z} - \bar{\xi}) \hat{g}(y + \eta, \bar{y} + \bar{\eta}, z - \eta, \bar{z} + \bar{\eta}) ,$$

where the hats are used to indicate functions that depend on both (y, \bar{y}) and (z, \bar{z}) , while functions depending only on (y, \bar{y}) shall be written without hats. The basic master fields are differential forms in an extended spacetime with coordinates $(x^M, z^\alpha, \bar{z}^\dot{\alpha})$, namely a one-form

$$\hat{A} = dx^M \hat{A}_M(x, z, \bar{z}; y, \bar{y}) + dz^\alpha \hat{A}_\alpha(x, z, \bar{z}; y, \bar{y}) + d\bar{z}^\dot{\alpha} \hat{A}_{\dot{\alpha}}(x, z, \bar{z}; y, \bar{y}) , \quad (2.6)$$

and a zero-form $\hat{\Phi} = \hat{\Phi}(x, z, \bar{z}; y, \bar{y})$.

The full higher-spin equations based on the above algebraic structures were first given in [1]. It can be shown that parity invariance and manifest Lorentz invariance restricts the possible interactions to two cases referred to as the minimal Type A and Type B models, in which the scalar field is even and odd under parity, respectively [9, 7]. The resulting master-field equations read

$$\hat{F} = \frac{i}{4} b_1 dz^\alpha \wedge dz_\alpha \hat{\Phi} \star \kappa + \frac{i}{4} (b_1)^\star d\bar{z}^\dot{\alpha} \wedge d\bar{z}_{\dot{\alpha}} \hat{\Phi} \star \bar{\kappa} , \quad (2.7)$$

$$\hat{D}\hat{\Phi} = 0 , \quad (2.8)$$

where $b_1 = 1$ and $b_1 = i$ in the Type A and Type B models, respectively, and the curvatures are defined as

$$\hat{F} = d\hat{A} + \hat{A} \star \hat{A} , \quad \hat{D}\hat{\Phi} = d\hat{\Phi} + [\hat{A}, \hat{\Phi}]_\pi , \quad (2.9)$$

with

$$[\hat{f}, \hat{g}]_\pi = \hat{f} \star \hat{g} - \hat{g} \star \pi(\hat{f}) , \quad (2.10)$$

and the functions κ and $\bar{\kappa}$ defined by

$$\kappa = \exp(iy^\alpha z_\alpha) , \quad \bar{\kappa} = \kappa^\dagger = \exp(-i\bar{y}^\dot{\alpha} \bar{z}_{\dot{\alpha}}) , \quad (2.11)$$

have the salient properties

$$\kappa \star \hat{f}(y, z) = \kappa \hat{f}(z, y) , \quad \hat{f}(y, z) \star \kappa = \kappa \hat{f}(-z, -y) . \quad (2.12)$$

$$\kappa \star \hat{f} \star \kappa = \pi(\hat{f}) . \quad (2.13)$$

In order to restrict to the minimal bosonic model one imposes further kinematic conditions

$$\tau(\hat{A}) = -\hat{A} , \quad \hat{A}^\dagger = -\hat{A} , \quad \tau(\hat{\Phi}) = \bar{\pi}(\hat{\Phi}) , \quad \hat{\Phi}^\dagger = \pi(\hat{\Phi}) , \quad (2.14)$$

where τ is the anti-automorphism

$$\tau(\hat{f}(y, \bar{y}, z, \bar{z})) = \hat{f}(iy, i\bar{y}, -iz, -i\bar{z}) , \quad \tau(\hat{f} \star \hat{g}) = \tau(\hat{g}) \star \tau(\hat{f}) , \quad (2.15)$$

and π is the involutive automorphism

$$\pi(\widehat{f}(y, \bar{y}, z, \bar{z})) = \widehat{f}(-y, \bar{y}, -z, \bar{z}) = \kappa \star \widehat{f} \star \kappa, \quad \pi(\widehat{f} \star \widehat{g}) = \pi(\widehat{f}) \star \pi(\widehat{g}). \quad (2.16)$$

The gauge transformations are given by

$$\delta_{\widehat{\epsilon}} \widehat{A} = \widehat{D}\widehat{\epsilon}, \quad \delta_{\widehat{\epsilon}} \widehat{\Phi} = -[\widehat{\epsilon}, \widehat{\Phi}]_{\pi}. \quad (2.17)$$

The rigid, *i.e.* x and Z -independent, gauge parameters form the Lie algebra

$$hs(4) = \{P(y, \bar{y}) : \tau(P) = P^{\dagger} = -P\}, \quad (2.18)$$

whose maximal finite-dimensional subalgebra is given by $SO(3, 2)$, with generators (A.3).

To analyze the master-field equations (2.7) and (2.8) formally, one starts from an “initial” condition

$$\Phi(x; y, \bar{y}) = \widehat{\Phi}|_{Z=0}, \quad A_M(x; y, \bar{y}) = \widehat{A}_M|_{Z=0}, \quad (2.19)$$

and a suitable gauge condition on the internal connection, such as [6]

$$\widehat{A}_{\alpha}|_{\Phi=0} = 0, \quad (2.20)$$

and proceeds by obtaining the Z -dependence of the fields by integrating the “internal” constraints, *viz.*

$$\widehat{D}_{\alpha} \widehat{\Phi} = 0, \quad \widehat{F}_{\alpha\beta} = -\frac{ib_1}{2} \epsilon_{\alpha\beta} \widehat{\Phi} \star \kappa, \quad \widehat{F}_{\alpha\dot{\alpha}} = 0, \quad \widehat{F}_{\alpha M} = 0, \quad (2.21)$$

perturbatively in a Φ -expansion, denoted by

$$\widehat{\Phi} = \Phi + \sum_{n=2}^{\infty} \widehat{\Phi}^{(n)}, \quad \widehat{A}_{\alpha} = \sum_{n=1}^{\infty} \widehat{A}_{\alpha}^{(n)}, \quad \widehat{A}_M = A_M + \sum_{n=1}^{\infty} \widehat{A}_M^{(n)}. \quad (2.22)$$

The x -space constraints $\widehat{F}_{MN} = 0$ and $\widehat{D}_M \widehat{\Phi} = 0$ can then be shown to be perturbatively equivalent to $\widehat{F}_{MN}|_{Z=0} = 0$ and $\widehat{D}_M \widehat{\Phi}|_{Z=0} = 0$, that is

$$F_{MN} = -\sum_{p=1}^{\infty} \sum_{m+n=p} [\widehat{A}_M^{(m)}, \widehat{A}_N^{(n)}]_{\star}, \quad D_M \Phi = -\sum_{p=2}^{\infty} \sum_{m+n=p} [\widehat{A}_M^{(m)}, \widehat{\Phi}^{(n)}]_{\pi}, \quad (2.23)$$

where $F_{MN} = 2\partial_{[M} A_{N]} + [A_M, A_N]_{\star}$ and $D_M \Phi = \partial_M \Phi + [A_M, \Phi]_{\pi}$. These equations constitute a perturbatively Cartan-integrable system in x -space provided that the *full* Z -dependence is included at each order in Φ .

Since (2.23) are written entirely in terms of differential forms they are manifestly diffeomorphism invariant. In fact, they are invariant under homotopy transformations, whereby coordinate directions can be added and removed without affecting the physical content. Thus, in case x -space is homotopic to a four-dimensional space-time manifold with coordinates x^{μ} ($\mu = 0, 1, 2, 3$), then one can without loss of generality formulate (2.23) directly on this four-manifold.

2.2 THE SPACE-TIME FIELD EQUATIONS

In order to obtain the physical field equations on generally covariant form, one first has to Lorentz covariantise (2.23). To this end, one first identifies the full Lorentz generators acting on the hatted master fields as follows

$$\widehat{M}_{\alpha\beta} = \widehat{M}_{\alpha\beta}^{(0)} + \frac{1}{2} \{\widehat{S}_\alpha, \widehat{S}_\beta\}_* , \quad (2.24)$$

where $\widehat{M}_{\alpha\beta}^{(0)} = y_\alpha y_\beta - z_\alpha z_\beta$, and

$$\widehat{S}_\alpha = z_\alpha - 2i\widehat{A}_\alpha . \quad (2.25)$$

One can show that [2]

$$\delta_L \widehat{\Phi} \equiv -[\widehat{\epsilon}_L, \widehat{\Phi}]_\pi = -[\widehat{\epsilon}_0, \widehat{\Phi}]_* , \quad (2.26)$$

$$\delta_L \widehat{A}_\alpha \equiv \widehat{D}_\alpha \widehat{\epsilon}_L = -[\widehat{\epsilon}_0, \widehat{A}_\alpha]_* + \Lambda_\alpha^\beta \widehat{A}_\beta , \quad (2.27)$$

$$\delta_L \widehat{A}_\mu \equiv \widehat{D}_\mu \widehat{\epsilon}_L = -[\widehat{\epsilon}_0, \widehat{A}_\mu]_* + \left(\frac{1}{4i} \partial_\mu \Lambda^{\alpha\beta} \widehat{M}_{\alpha\beta} - \text{h.c.} \right) , \quad (2.28)$$

where $\widehat{\epsilon}_L = \frac{1}{4i} \Lambda^{\alpha\beta}(x) \widehat{M}_{\alpha\beta} - (\text{h.c.})$ are the full parameters, and $\widehat{\epsilon}_0 = \frac{1}{4i} \Lambda^{\alpha\beta} \widehat{M}_{\alpha\beta}^{(0)} - (\text{h.c.})$ are the parameters of canonical Lorentz transformations of the Y and Z oscillators. The canonically transforming component fields are thus obtained by Y and Z -expansion of \widehat{A}_α , $\widehat{\Phi}$ and $\widehat{A}_\mu - (\frac{1}{4i} \omega_\mu^{\alpha\beta} \widehat{M}_{\alpha\beta} - \text{h.c.})$ where $\omega_\mu^{\alpha\beta}$ is the Lorentz connection with $\delta_L \omega_\mu^{\alpha\beta} = \partial_\mu \Lambda^{\alpha\beta} + \Lambda^{\alpha\gamma} \omega_\mu^{\gamma\beta} + \Lambda^{\beta\gamma} \omega_\mu^{\gamma\alpha}$ (related conventions are given in Appendix A). Hence, introducing

$$\omega_\mu = \frac{1}{4i} \omega_\mu^{\alpha\beta} M_{\alpha\beta} - \text{h.c.} , \quad M_{\alpha\beta} = y_\alpha y_\beta , \quad (2.29)$$

$$\widehat{K}_\mu = \frac{1}{4i} \omega_\mu^{\alpha\beta} (\widehat{M}_{\alpha\beta} - M_{\alpha\beta}) - \text{h.c.} , \quad (2.30)$$

and using the gauge condition (2.20) to simplify $K_\mu = \widehat{K}_\mu|_{Z=0}$, the Lorentz covariant decomposition of the master-gauge field A_μ reads

$$A_\mu = e_\mu + \omega_\mu + W_\mu + K_\mu , \quad (2.31)$$

$$K_\mu = i\omega_\mu^{\alpha\beta} (\widehat{A}_\alpha \star \widehat{A}_\beta)|_{Z=0} - \text{h.c.} \quad (2.32)$$

where e_μ is the vielbein¹

¹The vielbein here is given in the higher-spin frame, where the torsion is non-vanishing. A discussion of the Einstein frames is given in [25].

$$e_\mu = \frac{1}{2i} e_\mu^{\alpha\dot{\alpha}} y_\alpha \bar{y}_{\dot{\alpha}}, \quad e_\mu^{\alpha\dot{\alpha}} = -\frac{\lambda}{2} (\sigma_a)^{\alpha\dot{\alpha}} e_\mu^a, \quad (2.33)$$

and $W_\mu = W_\mu(y, \bar{y})$ contains the higher-spin gauge fields. Indeed, inserting (2.31) into (2.23), the explicit Lorentz connections cancel, and one obtains a manifestly Lorentz covariant and space-time diffeomorphism invariant set of constraints [2, 6].

To describe the higher-spin gauge symmetries in a generally space-time covariant fashion, one proceeds using a *weak-field approximation* in which the higher-spin gauge fields, the scalar field and the Weyl tensors, including that of spin 2, are treated as weak fields, while *no approximation is made for the vielbein and the Lorentz connection*. It can then be shown that $\widehat{D}_\mu \widehat{\Phi}|_{Z=0} = 0$ contains the field equation for the physical scalar field

$$\phi = \Phi|_{Y=0}, \quad (2.34)$$

and that $\widehat{F}_{\mu\nu}|_{Z=0} = 0$ contains the field equation for the metric

$$g_{\mu\nu} = e_\mu^a e_{\nu a} = -2\lambda^{-2} e_\mu^{\alpha\dot{\alpha}} e_{\nu\alpha\dot{\alpha}}, \quad (2.35)$$

and a set of physical higher-spin fields given by the doubly-traceless symmetric tensor fields of rank $s = 4, 6, \dots$, given by

$$\phi_{\mu_1\mu_2\dots\mu_s} = 2i e_{(\mu_1}^{\alpha_1\dot{\alpha}_1} \dots e_{\mu_{s-1}}^{\alpha_{s-1}\dot{\alpha}_{s-1}} \frac{\partial^{s-1}}{\partial y^{\alpha_1} \dots \partial y^{\alpha_{s-1}}} \frac{\partial^{s-1}}{\partial \bar{y}^{\dot{\alpha}_1} \dots \partial \bar{y}^{\dot{\alpha}_{s-1}}} W_{\mu_s)}|_{Y=0}. \quad (2.36)$$

The *generally covariant physical field equations* are given up to second order in weak fields by

$$(\nabla^2 + 2\lambda^2) \phi = \left(\nabla^\mu P_\mu^{(2)} - \frac{i\lambda}{2} (\sigma^\mu)^{\alpha\dot{\alpha}} \frac{\partial}{\partial y^\alpha} \frac{\partial}{\partial \bar{y}^{\dot{\alpha}}} P_\mu^{(2)} \right)_{Y=0}, \quad (2.37)$$

$$(\sigma^{\mu\nu\rho})_\alpha^{\dot{\beta}} \mathcal{R}_{\nu\rho\dot{\beta}\dot{\alpha}} = (\sigma^{\mu\nu\rho})_\alpha^{\dot{\beta}} \left(\frac{\partial}{\partial \bar{y}^{\dot{\beta}}} \frac{\partial}{\partial \bar{y}^{\dot{\alpha}}} J_{\nu\rho}^{(2)} \right)_{Y=0}, \quad (2.38)$$

$$(\sigma_\mu^{\nu\rho})_{(\alpha_1}^{\dot{\beta}} \mathcal{F}_{\nu\rho\alpha_2\dots\alpha_{s-1})\dot{\beta}\dot{\alpha}_1\dots\dot{\alpha}_{s-1}} = (\sigma_\mu^{\nu\rho})_{(\alpha_1}^{\dot{\beta}} \left(\frac{\partial}{\partial y^{\alpha_2}} \dots \frac{\partial}{\partial y^{\alpha_{s-1}}} \frac{\partial}{\partial \bar{y}^{\dot{\beta}}} \dots \frac{\partial}{\partial \bar{y}^{\dot{\alpha}_{s-1}}} J_{\nu\rho}^{(2)} \right)_{Y=0} \quad (2.39)$$

where the source terms $P_\mu^{(2)}$ and $J_{\mu\nu}^{(2)}$ are provided in Appendix C; $\mathcal{R}_{\mu\nu\dot{\alpha}\dot{\beta}}$, which is defined in (A.7), is the self-dual part of the full $SO(3, 2)$ -valued curvature $d\Omega + \Omega \star \Omega$ with Ω given in (A.6); the higher-spin curvatures are defined by ($s = 4, 6, 8, \dots$)

$$\begin{aligned} \mathcal{F}_{\nu\rho\alpha_2\dots\alpha_{s-1}\dot{\beta}\dot{\gamma}\dot{\alpha}_2\dots\dot{\alpha}_{s-1}} &= 2\nabla_{[\nu} W_{\rho]\alpha_2\dots\alpha_{s-1}\dot{\beta}\dot{\gamma}\dot{\alpha}_2\dots\dot{\alpha}_{s-1}} \\ &\quad - (s-2)(\sigma_{\nu\rho}\sigma_\mu)_{(\alpha_2}^{\dot{\beta}} W_{\mu\alpha_3\dots\alpha_{s-1})\dot{\beta}\dot{\gamma}\dot{\delta}\dot{\alpha}_2\dots\dot{\alpha}_{s-1}} \\ &\quad - s(\sigma_\mu\sigma_{\nu\rho})_{(\dot{\beta}}^{\gamma} W_{\mu\gamma\alpha_2\alpha_3\dots\alpha_{s-1})\dot{\gamma}\dot{\alpha}_2\dots\dot{\alpha}_{s-1}) ; \end{aligned} \quad (2.40)$$

and the covariant derivatives in (2.37) and (2.40) are given by $\nabla = d + \omega$ with ω being the canonical Lorentz connection. The expression (2.40) contains the auxiliary gauge fields $W_{\mu\alpha(s-2)\dot{\alpha}(s)}$ and $W_{\mu\alpha(s-3)\dot{\alpha}(s+1)}$, of which the latter drops out from (2.39), while the former can be expressed explicitly in terms of the physical fields using curvature-dressed covariant derivatives $\widehat{\nabla}$, as explained in [6].

3 CONSTRUCTION OF SOLUTIONS

3.1 PERTURBATIVE SPACE-TIME APPROACH

In the first order of the weak-field expansion, it is consistent to truncate the higher-spin field equations to that of pure Einstein gravity with cosmological constant plus a free scalar field in a fixed background metric, *viz.*

$$\bar{R}_{\mu\nu} + 3\lambda^2 \bar{g}_{\mu\nu} = 0, \quad (\bar{\nabla}^2 + 2\lambda^2)\bar{\phi} = 0. \quad (3.1)$$

These equations are self-consistent, though they do not derive from an action. They provide a good approximation if $\bar{\phi}$, the spin-2 Weyl tensor $\bar{\Phi}_{\mu\nu\rho\sigma}$ and all their derivatives are small. In general, such solutions may describe spacetimes that are not asymptotically conformally flat in any region.

The higher-order corrections from the weak-field expansion yields a perturbative expansion in $\bar{\phi}$ and $\bar{\Phi}_{\mu\nu\rho\sigma}$ of the form

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \sum_{n=2}^{\infty} g_{\mu\nu}^{(n)}, \quad \phi = \bar{\phi} + \sum_{n=2}^{\infty} \phi^{(2)}, \quad W_{\mu} = \sum_{n=2}^{\infty} W_{\mu}^{(n)}. \quad (3.2)$$

The second-order corrections can be determined from (2.37), (2.38) and (2.39), where the higher-spin gauge fields and Weyl tensors do not contribute to $P_{\mu}^{(2)}$ and $J_{\mu\nu}^{(2)}$ to that order. In this sense, exact solutions to ordinary Einstein gravity with cosmological constant may be embedded into higher-spin gauge theory, albeit that finding the exact solutions in closed form amenable to studies of salient properties is highly non-trivial. Moreover, there are subtleties related to boundary conditions as well as the convergence of the perturbative expansion.

The higher-spin field equation (2.39) reads ($s = 4, 6, \dots$)

$$\bar{\mathcal{F}}_{\mu_1 \dots \mu_s} = \mathcal{O}((\text{weak fields})^2), \quad (3.3)$$

where the Fronsdal-like operator $\bar{\mathcal{F}}_{\mu_1 \dots \mu_s}$ is covariantised using the background metric $\bar{g}_{\mu\nu}$. The first-order truncation of (3.3) is not *self-consistent*, *i.e.* incompatible with linearized higher-spin gauge symmetry, unless $\bar{\Phi}_{\mu\nu\rho\sigma} = 0$, *i.e.* $\bar{g}_{\mu\nu} = g_{(0)\mu\nu}$ where the

subscript (0) denotes the AdS_4 background with curvature given by (A.12). Thus, the proper way to include higher-spin fields $\bar{\phi}_{\mu_1 \dots \mu_s}$ into the first order is to solve

$$\bar{\mathcal{F}}_{\mu_1 \dots \mu_s} = 0 , \quad (3.4)$$

including all gauge artifacts, and impose the gauge conditions order-by-order in weak-field expansion using the fully consistent higher-spin field equation (2.39).

In order to switch on higher-spin fields, it is therefore convenient to consider solutions in which the full Weyl zero-form Φ asymptotes to zero in some region of spacetime. We shall refer to such solutions as *asymptotically Weyl-flat solutions*.

The perturbative approach to these solutions is self-consistent in the sense that the linearized twisted-adjoint zero-form $C = C(x; y, \bar{y})$ obeying $dC + [\Omega_{(0)}, C]_\pi = 0$, where $\Omega_{(0)}$ is the AdS_4 connection, vanishes on the boundary. To demonstrate this, one writes the linearized equation as

$$\nabla_{(0)\mu} C + \frac{\lambda}{2i} e_\mu^a \{P_a, C\}_\star = 0 , \quad (3.5)$$

and expands C and (3.5) in y and \bar{y} , which shows that C contains linearized Weyl tensors $C_{a(s),b(s)}$ obeying ($s = 2, 4, \dots$)

$$e_{(0)}^a \wedge e_{(0)}^b \wedge e_{(0)}^c \nabla_{(0)a} C_{b\mu(s-1),c\nu(s-1)} = 0 , \quad \nabla_{(0)}^\rho C_{\rho\mu(s-1),\nu(s)} = 0 . \quad (3.6)$$

These equations are in fact valid in AdS_D . The resulting mass-shell condition reads

$$[\nabla_{(0)}^2 + 2(s + D - 3)\lambda^2] C_{\mu(s),\nu(s)} = 0 . \quad (3.7)$$

Setting $s = 0$ one obtains formally the correct scalar-field equation,

$$(\nabla_{(0)}^2 + 2(D - 3)\lambda^2)\varphi = 0 , \quad (3.8)$$

where $\varphi = C|_{Y=0}$ in $D = 4$. Splitting $x^\mu \rightarrow (x^i, r)$, and using Poincaré coordinates,

$$ds_{(0)}^2 = \frac{1}{\lambda^2 r^2} (dr^2 + dx^2) , \quad \Gamma_{(0)ij}^r = \frac{1}{r} \eta_{ij} , \quad \Gamma_{(0)ir}^j = -\frac{1}{r} \delta_i^j , \quad \Gamma_{(0)rr}^r = -\frac{1}{r} , \quad (3.9)$$

one finds that the component fields $C_{i(s),j(t)r(s-t)}$ with $s \geq t \geq 1$, and where the indices are curved, are given by curls of $C_{i(s),r(s)}$, taken using $r\partial_j$, that in their turn obey

$$\left[\left(\partial_r + \frac{2-D}{r} \right) \left(\partial_r + \frac{2s}{r} \right) + \frac{s^2 + (D-1)s + 2(D-3)}{r^2} + \eta^{ij} \partial_i \partial_j \right] C_{i(s),r(s)} = 0 . \quad (3.10)$$

Thus, the linearized spin- s Weyl tensor $C_{a(s),b(s)}$ consists of two sectors $C_{a(s),b(s)}^{(\pm)}$ with scaling behavior given by ($s = 0, 2, 4, \dots$)

$$C_{a(s),b(s)}^{(\pm)} \sim r^{\beta_s^{\pm}}, \quad \beta_s^{\pm} = s + \frac{D-1}{2} \pm \frac{D-5}{2}. \quad (3.11)$$

At higher orders of the weak-field expansion, and due to the higher-derivative interactions hidden in the \star -products in $\widehat{D}_\mu \widehat{\Phi}|_{Z=0} = 0$, the corrections to the spin- s Weyl tensor Φ may in principle contain lower-spin constructs with a total scaling weight less than β_s^- . We shall not analyze the nature of these corrections in any further detail here, but hope to return to this interesting issue in a future work.

Another non-local effect induced via the Z -space dependence, is that the Lorentz covariantisations in K_μ , defined in (2.32), may remain finite in the asymptotic region, despite the naive expectation that the scaling of $(\widehat{A}_{(\alpha} \star \widehat{A}_{\beta)})|_{Z=0}$, which is of order Φ^2 , should over-power that of $\omega_\mu^{\alpha\beta}$. While this is a challenging problem to address in its generality, we shall find that already the relatively simple case of the $SO(3,1)$ -invariant asymptotically Weyl-flat solution exhibits an interesting phenomenon whereby K_μ generates a finite Weyl rescaling and contorsion in the asymptotic region.

Clearly, the weak-field expansion, which is naturally geared towards dressing up solutions to (3.1), is going to be far from efficient in dealing with general asymptotically Weyl-flat solutions. Especially when the starting point is not a solution to (3.1), it is appropriate to develop an alternative approach to solving the basic master-field equations (2.7) and (2.8) in which one makes a maximum use of the fact that the *local* x -dependence is a gauge choice. As we shall see next, the Z -space approach indeed does exploit this fact, and provides a powerful framework for finding exact solutions to higher-spin field equations.

3.2 THE Z -SPACE APPROACH

In order to construct solutions one may consider the Z -space approach [24] in which the constraints in spacetime, *viz.*

$$\widehat{F}_{\mu\nu} = 0, \quad \widehat{F}_{\mu\alpha} = 0, \quad \widehat{D}_\mu \widehat{\Phi} = 0, \quad (3.12)$$

are integrated in simply connected space-time regions given the space-time zero-forms at a point p ,

$$\widehat{\Phi}' = \widehat{\Phi}|_p, \quad \widehat{A}'_\alpha = \widehat{A}_\alpha|_p, \quad (3.13)$$

and expressed explicitly as

$$\widehat{A}_\mu = \widehat{L}^{-1} \star \partial_\mu \widehat{L}, \quad \widehat{A}_\alpha = \widehat{L}^{-1} \star (\widehat{A}'_\alpha + \partial_\alpha) \widehat{L}, \quad \widehat{\Phi} = \widehat{L}^{-1} \star \widehat{\Phi}' \star \pi(\widehat{L}), \quad (3.14)$$

where $\widehat{L} = \widehat{L}(x, z, \bar{z}; y, \bar{y})$ is a gauge function, and

$$\widehat{L}|_p = 1, \quad \partial_\mu \widehat{A}'_\alpha = 0, \quad \partial_\mu \widehat{\Phi}' = 0. \quad (3.15)$$

The remaining constraints in Z -space, *viz.*

$$\widehat{F}'_{\alpha\beta} \equiv 2\partial_{[\alpha} \widehat{A}'_{\beta]} + [\widehat{A}'_\alpha, \widehat{A}'_\beta]_\star = -\frac{ib_1}{2} \epsilon_{\alpha\beta} \widehat{\Phi}' \star \kappa, \quad (3.16)$$

$$\widehat{F}'_{\alpha\dot{\beta}} \equiv \partial_\alpha \widehat{A}'_{\dot{\beta}} - \partial_{\dot{\beta}} \widehat{A}'_\alpha + [\widehat{A}'_\alpha, \widehat{A}'_{\dot{\beta}}]_\star = 0, \quad (3.17)$$

$$\widehat{D}'_\alpha \widehat{\Phi}' \equiv \partial_\alpha \widehat{\Phi}' + \widehat{A}'_\alpha \star \widehat{\Phi}' + \widehat{\Phi}' \star \pi(\widehat{A}'_\alpha) = 0, \quad (3.18)$$

must then be solved given an initial condition

$$C'(y, \bar{y}) = \widehat{\Phi}'|_{Z=0}, \quad (3.19)$$

and fixing a suitable gauge for the internal connection. The natural choice is

$$\widehat{A}'_\alpha|_{C'=0} = 0, \quad (3.20)$$

whose compatibility with (2.20) requires

$$\widehat{L}|_{C'=0} = L(x; y, \bar{y}), \quad (3.21)$$

that is, the gauge function cannot depend explicitly on the Z -space coordinates. In what follows, we shall assume that

$$\widehat{L} = L(x; y, \bar{y}). \quad (3.22)$$

The gauge fields can then be obtained from (2.31), *viz.*

$$e_\mu + \omega_\mu + W_\mu = L^{-1} \partial_\mu L - K_\mu, \quad (3.23)$$

where

$$K_\mu = \widehat{K}_\mu|_{Z=0}, \quad \widehat{K}_\mu = i\omega_\mu^{\alpha\beta} L^{-1} \star \widehat{A}'_\alpha \star \widehat{A}'_\beta \star L. \quad (3.24)$$

Hence, the gauge fields, including the metric, can be obtained algebraically without having to solve any differential equations in spacetime.

The local representatives in two overlapping simply connected regions with gauge functions L and \tilde{L} , are related on the overlap via a gauge transformation with transition function

$$g = L^{-1} \star \tilde{L}. \quad (3.25)$$

The resulting transformations of the physical fields, *viz.*

$$\begin{aligned} \tilde{e}_\mu + \tilde{\omega}_\mu + \widetilde{W}_\mu &= g^{-1} \star [e_\mu + \omega_\mu + W_\mu + K_\mu - g \star (g^{-1} \star \widehat{K}_\mu \star g)|_{Z=0} \star g^{-1} + \partial_\mu] \star g \\ \tilde{\phi} &= (g^{-1} \star \widehat{\Phi} \star g)|_{Z=0}. \end{aligned} \quad (3.26) \quad (3.27)$$

which are a manifest symmetry of the full constraints in x and Z -space, are a non-trivial symmetry from the point of view of the generally covariant space-time field equations contained in $\hat{F}_{\mu\nu}|_{Z=0} = 0$ and $\hat{D}_\mu\hat{\Phi}|_{Z=0} = 0$ (whose general covariance corresponds to field-dependent gauge parameters).

To describe asymptotically Weyl-flat solutions, one chooses the gauge function to be a parameterization of the coset

$$L(x; y, \bar{y}) \in \frac{SO(3, 2)}{SO(3, 1)}. \quad (3.28)$$

By construction, its Maurer-Cartan form

$$L^{-1} \star dL = \Omega_{(0)} = e_{(0)} + \omega_{(0)}, \quad (3.29)$$

which means that the gauge fields defined by (3.23) are given asymptotically by the AdS_4 vacuum plus corrections from K_μ . The latter are higher order in the C' -expansion², but may nonetheless contribute in the asymptotic region to the leading dependence on the radial coordinate defined in (3.9). A concrete exemplification of this subtlety is provided by the asymptotic behavior of the scale factors in the $SO(3, 1)$ -invariant solution discussed at the end of Section 4.3.

3.3 ON REGULAR, SINGULAR AND PSEUDO-SINGULAR INITIAL CONDITIONS

The perturbative approach to (3.16)–(3.18) yields a solution of the form

$$\hat{\Phi}' = C' + \sum_{n=2}^{\infty} \hat{\Phi}'^{(n)}, \quad \hat{A}'_\alpha = \sum_{n=1}^{\infty} \hat{A}'_\alpha^{(n)}, \quad (3.30)$$

where the superscripts indicate the order in C' . We shall refer to C' as a *regular initial condition* provided that its \star -product self-compositions, *viz.*

$$C'_{(2n)} = (C' \star \pi(C'))^{\star n}, \quad (3.31)$$

$$C'_{(2n+1)} = (C' \star \pi(C'))^{\star n} \star C', \quad (3.32)$$

are regular functions. The second-order corrections (C.4) and (C.5) contain the \star -product composition

$$(C'(-tz, \bar{y})e^{ityz}) \star C'(y, \bar{y}) = \kappa \star (C'(-ty, \bar{y})e^{i(1-t)yz}) \star C'(y, \bar{y}), \quad (3.33)$$

²Using the gauge function (4.1), the spin- s sector in $C(x; y, \bar{y}) = L^{-1}(x) \star C'(y, \bar{y}) \star L(x)$ scales like $h^{2(1+s)}$, *i.e.* C' contains the regular boundary data scaling like $r^{\beta_s^-}$ in the notation of (3.11) with $r \sim h^2$.

where t is the auxiliary integration parameter used to present the first-order correction (C.3) to the internal connection. If C' is regular, then it follows that (3.33) is a regular function of (Y, Z) with t -dependent coefficients that are analytic at $t = 1$. We shall assume analyticity also at $t = 0$, where (3.33) involves only anti-holomorphic contractions resulting in a “softer” composition that should not blow up. Under these assumptions there exists an open contour from $t = 0$ to $t = 1$ along which (3.33) is analytic, that can then be used in (C.3) to produce a well-defined second order correction. This argument extends to higher orders of perturbation theory, and hence regular initial data yields perturbative corrections that can be presented using open integration contours [6].

If C' is not regular, and (3.33) has an *isolated* singularity at $t = 1$, we shall refer to C' as a *singular initial condition*. In this case, a well-defined perturbative expansion can be obtained by circumventing the singularity by using a closed contour γ as follows,

$$d' \oint_{\gamma} \frac{dt}{2\pi i} \log \left(\frac{t}{1-t} \right) Z^{\underline{\beta}} f_{\underline{\beta}}(tZ) = dZ^{\underline{\alpha}} f_{\underline{\alpha}}(Z) , \quad (3.34)$$

and

$$d' \oint_{\gamma} \frac{dt}{2\pi i} \left(t \log \frac{t}{1-t} - 1 \right) Z^{\underline{\alpha}} dZ^{\underline{\beta}} f_{\underline{\alpha}\underline{\beta}}(tZ) = \frac{1}{2} dZ^{\underline{\alpha}} \wedge dZ^{\underline{\beta}} f_{\underline{\alpha}\underline{\beta}}(Z) , \quad (3.35)$$

where $Z^{\underline{\alpha}} = (z^{\alpha}, \bar{z}^{\dot{\alpha}})$, $d' = dZ^{\underline{\alpha}} \partial_{\underline{\alpha}}$, $d' f_1 = d' f_2 = 0$ and γ encircles the branch cut from $t = 0$ to $t = 1$. The resulting closed-contour presentation of the perturbative solution can of course also be used in the case of regular initial data, in which case γ can be collapsed onto the branch-cut as to reproduce the open-contour presentation.

Singular initial conditions may arise from imposing symmetry conditions on C' , as will be exemplified in Section 5.3 in cases where the symmetry refers to unbroken space-time isometries. Here the initial conditions are parameterized elementary functions of the oscillators, *e.g.* combinations of exponentials of the bilinear translation generator P_a contracted with fixed vectors, that become singular for particular choices of the parameters.

Another interesting type of irregular initial data arises in the five-dimensional and seven-dimensional higher-spin models based on spinor oscillators [10, 11, 12] and the D -dimensional model based on vector oscillators. Here the initial conditions are regular functions multiplied with *singular projectors* whose role is to gauge internal symmetries in oscillator space in order that the master-field equations contain dynamics. These symmetries are generated by bilinear oscillator constructs, K , and the singular projectors are special functions with auxiliary integral representations, given schematically by $\int_0^1 ds f(s) e^{sK}$, such that the analogs of the self-compositions (3.31) and (3.32) now contain *logarithmic divergencies arising in corners of the auxiliary integration domain*. Due to the projection property, these divergencies can be

factored out and written as [4]

$$\int_0^1 \frac{ds}{1-s} g(s) , \quad (3.36)$$

where $g(s)$ is analytic and non-vanishing at $s = 1$. One can now argue that the perturbative expansion gives rise to analogs of (3.33) resulting in regular functions of (Y, Z) with t -dependent coefficients involving pre-factors of the form

$$\int_0^1 \frac{ds}{1-ts} g(s) , \quad (3.37)$$

that are logarithmically divergent at $t = 1$. Thus, the open-contour presentation results in well-defined second-order perturbations containing pre-factors of the form

$$\int_0^1 \int_0^1 \frac{ds dt}{1-st} g(s) , \quad (3.38)$$

while the closed-contour presentation, which requires analyticity on closed curve encircling $t = 1$, does not apply since the singularity at $t = 1$ is not isolated. We shall therefore refer to initial conditions of this type as *pseudo-singular initial conditions*.

The above analysis indicate that the (pseudo-)singular nature of initial conditions is an artifact of the naive application of the \star -product algebra to (3.32), while the actual perturbative expansion in Z -space involves a point-splitting mechanism that softens the divergencies. We plan to return with a more conclusive report on these important issues in a forthcoming paper.

3.4 ZERO-FORM CURVATURE INVARIANTS

In unfolded dynamics, the local degrees of freedom are dual to the twisted-adjoint element $C'(y, \bar{y})$ defined by (3.19), consisting of all gauge-covariant derivatives of the physical fields evaluated at a point in spacetime. At the linearized level, C' is gauge covariant, which means that if $C' = g^{-1} \star \tilde{C}' \star \pi(g)$, with g a group element generated by $hs(4)$, then C' and \tilde{C}' give rise to gauge-equivalent solutions.

To distinguish between gauge-inequivalent solutions at the full level, we propose the following invariants

$$\mathcal{C}_{2p}^\pm = \mathcal{N}_\pm \hat{\mathcal{T}}r_\pm \hat{\mathcal{C}}_{2p} , \quad (3.39)$$

for $p = 1, 2, \dots$, where

$$\hat{\mathcal{C}}_{2p} = \underbrace{\hat{\Phi} \star \kappa \star \dots \star \hat{\Phi} \star \kappa}_{2p \text{ times}} = (\hat{\Phi} \star \pi(\hat{\Phi}))^{\star p} ; \quad (3.40)$$

the full traces are defined by

$$\widehat{Tr}_+ \widehat{f} = \int \frac{d^2y d^2\bar{y} d^2z d^2\bar{z}}{(2\pi)^4} \widehat{f}(y, \bar{y}, z, \bar{z}), \quad \widehat{Tr}_- \widehat{f} = \widehat{Tr}_+(\widehat{f} \star \kappa \bar{\kappa}); \quad (3.41)$$

and the normalizations \mathcal{N}_\pm are given by

$$\mathcal{N}_- = 1, \quad \mathcal{N}_+ = \frac{1}{\mathcal{V}}, \quad (3.42)$$

where \mathcal{V} is the volume of Z -space. As separate components of Φ vary over spacetime, the net effect is that the invariants \mathcal{C}_{2p}^\pm remain constant, though they may diverge for specific solutions.

Let us motivate the above definitions. To begin with, it follows from (2.5) and (2.12) that the full traces obey

$$\widehat{Tr}_\pm(\widehat{f}(Y, Z) \star \widehat{g}(Y, Z)) = \widehat{Tr}_\pm(\widehat{g}(\pm Y, \pm Z) \star \widehat{f}(Y, Z)), \quad (3.43)$$

and

$$\widehat{Tr}_- f(Y) = Tr_- f(Y), \quad (3.44)$$

where the reduced traces $Tr_\pm f(Y)$ are defined by³

$$Tr_+ f(Y) = \int \frac{d^4Y}{(2\pi)^2} f(Y), \quad Tr_- f(Y) = f(0). \quad (3.45)$$

Moreover, from $\widehat{D}_\mu \widehat{\Phi} = 0$, which is equivalent to $\partial_\mu(\widehat{\Phi} \star \kappa) = -[\widehat{A}_\mu, \widehat{\Phi} \star \kappa]$, it follows that

$$\partial_\mu \widehat{\mathcal{C}}_q = -[\widehat{A}_\mu, \widehat{\mathcal{C}}_q]_\star, \quad \widehat{\mathcal{C}}_q = (\widehat{\Phi} \star \kappa)^{\star q}, \quad (3.46)$$

for any positive integer q , which together with (3.43) and the fact that \widehat{A}_μ is an even function of all the oscillators implies that formally ($q = 1, 2, \dots$)

$$d\widehat{Tr}_\pm \widehat{\mathcal{C}}_q = 0. \quad (3.47)$$

This property can be made manifest by going to primed basis using (3.14).

Expanding perturbatively,

$$\widehat{\mathcal{C}}_q = (\Phi \star \kappa)^{\star q} + \sum_{n=q+1}^{\infty} \widehat{\mathcal{C}}_q^{(n)}, \quad (3.48)$$

and taking $q = 2p$, one finds that the leading contribution to the charges are given by

$$\mathcal{C}_{2p}^{-(2p)} = Tr_-(\Phi \star \pi(\Phi))^{\star p}, \quad (3.49)$$

³The odd trace Tr_- for the Weyl algebra was originally introduced in [26].

where we have used (3.44), and

$$\mathcal{C}_{2p}^{+(2p)} = Tr_+(\Phi \star \pi(\Phi))^{\star p} , \quad (3.50)$$

where we have formally factored out and cancelled the volume of Z -space. For $q = 2p + 1$ one finds that the leading order contribution to $\hat{Tr}_{\pm} \hat{\mathcal{C}}_{2p+1}$ diverges like $\int d^2z$ or $\int d^2\bar{z}$, whose regularization we shall not consider here. The higher-order corrections to the charges, may require additional prescriptions, and, as already mentioned, they need not be well-defined in general.

The form of the leading contribution $\mathcal{C}_{2p}^{-(2p)}$ suggests that the full charges \mathcal{C}_{2p}^- are well-defined for general regular initial data. Moreover, these charges can be rewritten on a more suggestive form using (2.7), where $dz^\alpha \wedge dz_\alpha = -2d^2z$, and $\pi(\hat{\Phi}) = \bar{\pi}(\hat{\Phi})$, which imply

$$\hat{F} \star \hat{F} \star \hat{\mathcal{C}}_q = -\frac{1}{2} d^2z \wedge d^2\bar{z} \hat{\mathcal{C}}_{q+2} \star \kappa \bar{\kappa} , \quad (3.51)$$

so that

$$\mathcal{C}_{2p}^- = -Tr_+ \left[\frac{1}{2\pi^2} \int \hat{F} \star \hat{F} \star (\hat{\Phi} \star \pi(\hat{\Phi}))^{\star(p-1)} \right] , \quad (3.52)$$

which can be used to show that the charges can be written as total derivatives in Z order by order in the Φ -expansion.

4 AN $SO(3, 1)$ -INVARIANT SOLUTION

4.1 THE ANSATZ

To find $SO(3, 1)$ -invariant solutions we use the Z -space approach based on (3.14). It is convenient to use a Lorentz-covariant parameterization of the gauge function, *viz.* [24]

$$L(x; y, \bar{y}) = \exp[i f(x^2) x^{\alpha\dot{\alpha}} y_\alpha \bar{y}_{\dot{\alpha}} + r(x^2)]$$

with $x^{\alpha\dot{\alpha}} = (\sigma_a)^{\alpha\dot{\alpha}} x^a$ and $x^2 = x^a x_a$. The function $r(x^2)$ is fixed by demanding $\tau(L) = L^{-1}$, so that $L \in SO(3, 2)$ and hence $L^{-1} \star dL$ describes the AdS_4 vacuum solution⁴. Using $L^{-1} = (1 - f^2 x^2)^2 \exp[-if(x^2) x^{\alpha\dot{\alpha}} y_\alpha \bar{y}_{\dot{\alpha}} - r(x^2)]$ one finds $r = \log(1 - f^2 x^2)$. A convenient choice of f is [24]

$$L(x; y, \bar{y}) = \frac{2h}{1+h} \exp \left[\frac{i\lambda x^{\alpha\dot{\alpha}} y_\alpha \bar{y}_{\dot{\alpha}}}{1+h} \right] , \quad \lambda^2 x^2 < 1 , \quad (4.1)$$

⁴To exhibit $L \in SO(3, 2)$ one can write $L(x; y, \bar{y}) = \exp_*(i \frac{\operatorname{artanh} \sqrt{\frac{1-h}{1+h}}}{\sqrt{1-h^2}} \lambda x^{\alpha\dot{\alpha}} y_\alpha \bar{y}_{\dot{\alpha}})$ where $\exp_* A = 1 + A + \frac{1}{2} A \star A + \dots$

where

$$x_{\alpha\dot{\alpha}} = (\sigma^a)_{\alpha\dot{\alpha}} x_a, \quad h = \sqrt{1 - \lambda^2 x^2}, \quad x^2 = x^a x_a \quad (4.2)$$

corresponding to the vierbein and Lorentz connection

$$e_{(0)}{}^{\alpha\dot{\alpha}} = -\frac{\lambda(\sigma^a)_{\alpha\dot{\alpha}} dx_a}{h^2}, \quad \omega_{(0)}{}^{\alpha\beta} = -\frac{\lambda^2(\sigma^{ab})^{\alpha\beta} dx_a x_b}{h^2}, \quad (4.3)$$

that in turn gives

$$ds_{(0)}^2 = \frac{4dx^2}{(1 - \lambda^2 x^2)^2}, \quad (4.4)$$

which one identifies as the metric of AdS_4 spacetime with inverse radius λ given in stereographic coordinates. The inversion $x^\mu \rightarrow -x^\mu/(\lambda^2 x^2)$ maps the space-like regions $0 < \lambda^2 x^2 < 1$ and $\lambda^2 x^2 > 1$ into each other and the boundary $\lambda^2 x^2 = 1$ onto itself. The future and past time-like regions are mapped onto themselves, with distant past and future, where $\lambda^2 x^2 \rightarrow -\infty$, sent to the past and future light cones, respectively. The two space-like and the two time-like regions provide a global cover of AdS_4 spacetime, so that beyond the distant past and future lies the space-like region $\lambda^2 x^2 > 1$, as can be seen more explicitly using the global parametrization $ds^2 = -(1+r^2)dt^2 + (1+r^2)^{-1}dr^2 + r^2 d\Omega_2^2$ where $x^2 = 2(1 + \sin t\sqrt{1+r^2})^{-1}$. Thus, a global description can be obtained using the additional gauge function,

$$\tilde{L} = L(\tilde{x}; y, \bar{y}), \quad \lambda^2 \tilde{x}^2 < 1, \quad (4.5)$$

and declaring the overlap with $L(x; y, \bar{y})$ to be given by

$$\tilde{x}^a = -\frac{x^a}{\lambda^2 x^2}, \quad \lambda^2 x^2 = 1/(\lambda^2 \tilde{x}^2) < 0, \quad (4.6)$$

leading to the transition function $\tilde{L}^{-1} \star L$. The Z_2 -symmetry implies that the local representatives of the full solution will be given by the *same* functions, with $x^a \leftrightarrow \tilde{x}^a$, as we shall discuss in more detail in Section 4.3⁵.

A particular type of $SO(3, 1)$ -invariant solutions can be obtained by imposing

$$[\widehat{M}'_{\alpha\beta}, \widehat{\Phi}']_\pi = 0, \quad (4.7)$$

$$\widehat{D}'_\alpha \widehat{M}'_{\beta\gamma} = 0, \quad (4.8)$$

where $\widehat{M}'_{\alpha\beta}$ are the full Lorentz generators defined in (2.24) and given in the primed basis, and obeying (2.26)–(2.28). Eq. (4.7) combined with (2.26) imply that

$$\widehat{\Phi}' = \widehat{\Phi}'(u, \bar{u}), \quad (4.9)$$

⁵One can describe anti-de Sitter spacetime using the globally well-defined gauge function $L = (1 - \lambda^2 x^2) \exp i\lambda x^{\alpha\dot{\alpha}} y_\alpha \bar{y}_{\dot{\alpha}}$ where $\lambda^2 x^2 \neq 1$. The corresponding $SO(3, 1)$ -invariant solution has Lorentz connection and vierbein given by (4.61) and (4.62) with a^2 replaced by x^2 in (4.59). These expressions are ill-defined for $|x^2| > 1$, which means that a globally well-defined solution still requires another coordinate patch.

where

$$u = y^\alpha z_\alpha, \quad \bar{u} = u^\dagger = \bar{y}^{\dot{\alpha}} \bar{z}_{\dot{\alpha}}, \quad (4.10)$$

and $(\hat{\Phi}'(u, \bar{u}))^\dagger = \hat{\Phi}'(u, \bar{u})$. Moreover, from (4.8) combined with (2.27) and the τ -invariance condition on \hat{A}_α , it follows that

$$\hat{A}'_\alpha = z_\alpha A(u, \bar{u}), \quad \hat{S}'_\alpha = z_\alpha S(u, \bar{u}), \quad S = 1 - 2iA. \quad (4.11)$$

We next turn to the exact solution of the Z -space equations (3.16)–(3.18).

4.2 SOLUTION OF Z -SPACE EQUATIONS

The internal constraints $\hat{F}'_{\alpha\dot{\alpha}} = 0$ and $\hat{D}'_\alpha \hat{\Phi}' = 0$ are solved by

$$S(u, \bar{u}) = S(u), \quad \hat{\Phi}'(u, \bar{u}) = \frac{\nu}{b_1}, \quad (4.12)$$

where ν/b_1 is a real constant, so that ν is real in the Type A model and purely imaginary in the Type B model.

The remaining constraint on $\hat{F}'_{\alpha\beta}$, given by (3.16), now takes the form

$$[\hat{S}'^\alpha, \hat{S}'_\alpha]_\star = 4i(1 - \nu e^{iu}). \quad (4.13)$$

To solve this constraint, following [23], we use the integral representation

$$S(u) = \int_{-1}^1 ds m(s) e^{\frac{i}{2}(1+s)u}, \quad (4.14)$$

where the choice of contour is motivated by the relation

$$\begin{aligned} & (z_\alpha e^{\frac{i}{2}(1+s)u}) \star (z_\beta e^{\frac{i}{2}(1+s')u}) \\ &= \left(-i\epsilon_{\alpha\beta} - \frac{1}{4}[y - z + s'(y + z)]_\alpha [y + z + s(y - z)]_\beta \right) e^{\frac{i}{2}(1-ss')u}, \end{aligned} \quad (4.15)$$

which induces the map $(s, s') \mapsto -ss'$ from $[-1, 1] \times [-1, 1]$ to $[-1, 1]$. As a result (4.13) becomes

$$\int_{-1}^1 ds \int_{-1}^1 ds' \left[1 + \frac{i}{4}(1 - ss')u \right] m(s) m(s') e^{\frac{i}{2}(1-ss')u} = 1 - \nu e^{iu}, \quad (4.16)$$

which can be written as

$$\int_{-1}^1 dt g(t) \left[1 + \frac{i}{4}(1 - t)u \right] e^{\frac{i}{2}(1-t)u} = 1 - \nu e^{iu}, \quad (4.17)$$

where $g = m \circ m$ with \circ defined by [23]

$$(p \circ q)(t) = \int_{-1}^1 ds \int_{-1}^1 ds' \delta(t - ss') p(s) q(s') . \quad (4.18)$$

Replacing iu by $-2d/dt$ acting on the exponential and integrating by parts, we find

$$\int_{-1}^1 dt \left[g(t) + \frac{1}{2} ((1-t)g(t))' \right] e^{\frac{i}{2}(1-t)u} - \frac{1}{2} \left[(1-t)g(t) e^{\frac{i}{2}(1-t)u} \right]_{-1}^1 = 1 - \nu e^{iu} . \quad (4.19)$$

This can be satisfied by taking g to obey

$$g(t) + \frac{1}{2} ((1-t)g(t))' = \delta(1-t) , \quad g(-1) = -\nu , \quad (4.20)$$

with the solution

$$(m \circ m)(t) = g(t) = \delta(t-1) - \frac{\nu}{2}(1-t) . \quad (4.21)$$

Even and odd functions are orthogonal with respect to the \circ product, *i.e.*

$$p^{(\sigma)} \circ q^{(\sigma')} = \delta_{\sigma\sigma'} p^{(\sigma)} \circ q^{(\sigma')} , \quad p^{(\sigma)}(-t) = \sigma p^{(\sigma)}(t) , \quad \sigma = \pm 1 . \quad (4.22)$$

Therefore

$$(m^{(+)} \circ m^{(+)})(t) = I_0^{(+)}(t) - \frac{\nu}{2} , \quad (4.23)$$

$$(m^{(-)} \circ m^{(-)})(t) = I_0^{(-)}(t) + \frac{\nu}{2}t , \quad (4.24)$$

where

$$I_0^{(\pm)}(t) = \frac{1}{2} [\delta(1-t) \pm \delta(1+t)] . \quad (4.25)$$

One proceeds [23], by expanding $m^{(\pm)}(t)$ in terms of $I_0^{(\pm)}(t)$ and the functions ($k \geq 1$)

$$\begin{aligned} I_k^{(\sigma)}(t) &= [\text{sign}(t)]^{\frac{1}{2}(1-\sigma)} \int_{-1}^1 ds_1 \cdots \int_{-1}^1 ds_k \delta(t - s_1 \cdots s_k) \\ &= [\text{sign}(t)]^{\frac{1}{2}(1-\sigma)} \frac{\left(\log \frac{1}{t^2}\right)^{k-1}}{(k-1)!} , \end{aligned} \quad (4.26)$$

which obey the algebra ($k, l \geq 0$)

$$I_k^{(\sigma)} \circ I_l^{(\sigma)} = I_{k+l}^{(\sigma)} . \quad (4.27)$$

Thus, given a quantity

$$p^{(\sigma)}(t) = \sum_{k=0}^{\infty} p_k I_k^{(\sigma)}(t) , \quad (4.28)$$

and defining its symbol

$$\tilde{p}^{(\sigma)}(\xi) = \sum_{k=0}^{\infty} p_k \xi^k , \quad (4.29)$$

it follows from (4.27) that

$$(p^\sigma \circ q^\sigma)(\xi) = \tilde{p}^{(\sigma)}(\xi) \tilde{q}^{(\sigma)}(\xi) , \quad (4.30)$$

so that (4.23) and (4.24) become the algebraic equations

$$\begin{aligned} (\tilde{m}^{(+)}(\xi))^2 &= 1 - \frac{\nu}{2}\xi , \\ (\tilde{m}^{(-)}(\xi))^2 &= 1 + \frac{\nu}{2} \frac{\xi}{1 + \frac{1}{2}\xi} . \end{aligned} \quad (4.31)$$

Therefore

$$\begin{aligned} m^{(+)}(t) &= \pm \left[I^{(+)}(t) + q^{(+)}(t) \right] , & \tilde{q}^{(+)}(\xi) &= \sqrt{1 - \frac{\nu}{2}\xi} - 1 , \\ m^{(-)}(t) &= \pm \left[I^{(-)}(t) + q^{(-)}(t) \right] , & \tilde{q}^{(-)}(\xi) &= \sqrt{1 + \frac{\nu}{2} \frac{\xi}{1 + \frac{1}{2}\xi}} - 1 . \end{aligned} \quad (4.32)$$

The physical gauge condition (3.20), which requires

$$m(t)|_{\nu=0} = \delta(1+t) , \quad (4.33)$$

implies that

$$m(t) = m^{(+)}(t) - m^{(-)}(t) = \delta(1+t) + q(t) , \quad (4.34)$$

$$q(t) = q^{(+)}(t) - q^{(-)}(t) . \quad (4.35)$$

To obtain the functions $q^{(\pm)}(t)$ explicitly, we first expand

$$q^{(\pm)}(t) = [\text{sign}(t)]^{\frac{1}{2}(1-\sigma)} \sum_{k=1}^{\infty} \tilde{q}_k^{(\pm)} \frac{\left(\log \frac{1}{t^2}\right)^{k-1}}{(k-1)!} , \quad (4.36)$$

where the coefficients are related to the expansions of the symbols as

$$\tilde{q}^{(\pm)}(\xi) = \sum_{k=1}^{\infty} \tilde{q}_k^{(\pm)} \xi^k . \quad (4.37)$$

In the case of $q^{(+)}(t)$, expansion of $\sqrt{1 - \frac{\nu}{2}\xi} - 1$ yields

$$q^{(+)}(t) = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k+1} \left(-\frac{\nu}{2}\right)^{k+1} \frac{\left(\log \frac{1}{t^2}\right)^k}{k!} = -\frac{\nu}{4} {}_1F_1\left[\frac{1}{2}; 2; \frac{\nu}{2} \log \frac{1}{t^2}\right]. \quad (4.38)$$

In the case of $q^{(-)}(t)$, we begin by defining

$$q^{(-)}(t) = \text{sign}(t) \hat{q}(\log \frac{1}{t^2}). \quad (4.39)$$

Thus, from

$$\hat{Q}(\zeta) \equiv \int_0^\zeta d\zeta' \hat{q}(\zeta') = \sum_{k=1}^{\infty} \hat{q}_k^{(-)} \frac{\zeta^k}{k!}, \quad (4.40)$$

and $\xi^k = \int_0^\infty d\xi e^{-\zeta} \frac{(\xi\xi)^k}{k!}$, it follows that $\hat{q}^{(-)}(\xi) = \int_0^\infty d\xi e^{-\zeta} \hat{Q}(\xi\xi)$. This Laplace-type transformation can be inverted as

$$\hat{Q}(\zeta) = \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dz e^{\zeta z}}{2\pi i z} \hat{q}^{(-)}\left(\frac{1}{z}\right), \quad (4.41)$$

with

$$\gamma > \max \left\{ \text{Re} z_i : z_i \text{ pole or branch cut of } \frac{1}{z} \hat{q}^{(-)}\left(\frac{1}{z}\right) \right\}.$$

The function $q^{(-)}(t)$ is then obtained by differentiation with respect to ζ and the substitution $\zeta = \log \frac{1}{t^2}$, that is

$$q^{(-)}(t) = \text{sign}(t) \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dz e^{\zeta z}}{2\pi i} \hat{q}^{(-)}\left(\frac{1}{z}\right) \Big|_{\zeta=\log \frac{1}{t^2}}. \quad (4.42)$$

The contour can be closed around the branch-cut that goes from $z = -\frac{1}{2}$ to $z = -\frac{1+\nu}{2}$, and one finds

$$q^{(-)}(t) = \frac{\nu t}{4} {}_1F_1\left[\frac{1}{2}; 2; -\frac{\nu}{2} \log \frac{1}{t^2}\right]. \quad (4.43)$$

In summary, the internal solution is given by

$$\hat{\Phi}' = \frac{\nu}{b_1}, \quad (4.44)$$

$$\hat{A}'_\alpha = \frac{i}{2} z_\alpha \int_{-1}^1 dt q(t) e^{\frac{i}{2}(1+t)u}, \quad (4.45)$$

$$q(t) = -\frac{\nu}{4} \left({}_1F_1\left[\frac{1}{2}; 2; \frac{\nu}{2} \log \frac{1}{t^2}\right] + t {}_1F_1\left[\frac{1}{2}; 2; -\frac{\nu}{2} \log \frac{1}{t^2}\right] \right). \quad (4.46)$$

Expanding $\exp(\frac{itu}{2})$ results in integrals of the degenerate hypergeometric functions times t^p ($p = 0, 1, \dots$), which improve the convergence at $t = 0$. Thus \hat{A}_α is a formal power-series expansion in u with coefficients that are functions of ν that are well-behaved provided this is the case for the coefficient of u^0 . This is the case for ν in some finite region around $\nu = 0$, as we shall see next.

4.3 THE SOLUTION IN SPACETIME

Let us evaluate the physical fields in the two regions $\lambda^2 x^2 < 1$ and $\lambda^2 \tilde{x}^2 < 1$ using the two gauge functions (4.1) and (4.5). We first compute $\hat{\Phi} = L^{-1} \star \hat{\Phi}' \star \pi(L) = \frac{\nu}{b_1} L^{-1} \star L^{-1}$, with the result

$$\hat{\Phi} = \frac{\nu}{b_1} (1 - \lambda^2 x^2) \exp \left[-i \lambda x^{\alpha\dot{\alpha}} y_\alpha \bar{y}_{\dot{\alpha}} \right]. \quad (4.47)$$

This shows that the physical scalar field is given in the x^a -coordinate chart by

$$\phi(x) = \hat{\Phi}|_{Y=Z=0} = \frac{\nu}{b_1} (1 - \lambda^2 x^2), \quad \lambda^2 x^2 < 1, \quad (4.48)$$

while the Weyl tensors for spin $s = 2, 4, \dots$ vanish. Using instead \tilde{L} , the physical scalar field in the \tilde{x}^a -coordinate chart is given by

$$\tilde{\phi} = \nu (1 - \lambda^2 \tilde{x}^2), \quad \lambda^2 \tilde{x}^2 < 1. \quad (4.49)$$

As a result, the two scalar fields are related by a duality transformation in the overlap region

$$\tilde{\phi}(\tilde{x}) = \frac{\nu \phi(x)}{\phi(x) - \nu}, \quad \lambda^2 x^2 = (\lambda^2 \tilde{x}^2)^{-1} < 0. \quad (4.50)$$

Thus, if the transition takes place at $\lambda^2 x^2 = \lambda^2 \tilde{x}^2 = -1$, then the amplitude of the physical scalar never exceeds 2ν so the open-string theory of [17], which couples to the Weyl zero-form, is weakly coupled throughout the solution. We also note that a gauge-invariant characterization of the solution is provided by the invariants (3.39), that are given by

$$\mathcal{C}_{2p}^- = (\nu/b_1)^{2p}, \quad (4.51)$$

while \mathcal{C}_{2p}^+ diverges.

The gauge fields are defined by the decomposition (2.31) with $\hat{A}_\mu = L^{-1} \star \partial_\mu L = e_\mu^{(0)} + \omega_\mu^{(0)}$, that is

$$e_\mu + W_\mu = e_\mu^{(0)} + \omega_\mu^{(0)} - K_\mu, \quad (4.52)$$

with $e_\mu^{(0)}$ and $\omega_\mu^{(0)}$ denoting the AdS_4 vacuum, and

$$K_\mu = i\omega_\mu^{\alpha\beta} L^{-1} \star \hat{A}'_\alpha \star \hat{A}'_\beta \star L - \text{h.c.} , \quad (4.53)$$

which can be rewritten using (4.15) as

$$\begin{aligned} K_\mu &= \frac{1}{16i} \omega_\mu^{\alpha\beta} \int_{-1}^1 dt \int_{-1}^1 dt' q(t)q(t') \times \\ &\quad \times \left\{ \left[(1+t)(1+t') \frac{\partial^2}{\partial \rho^\alpha \partial \rho^\beta} - (1-t)(1-t') \frac{\partial^2}{\partial \tau^\alpha \partial \tau^\beta} \right] V(y, \bar{y}; \rho, \tau; tt') \right\} \Big|_{\rho=\tau=0} \\ &\quad - \text{h.c.} , \end{aligned} \quad (4.54)$$

where

$$V(y, \bar{y}; \rho, \tau; tt') = \left[L^{-1} \star e^{i\left(\frac{1-tt'}{2}u+\rho y+\tau z\right)} \star L \right] \Big|_{Z=0} . \quad (4.55)$$

Using (B.6), this quantity can be expressed as

$$V(y, \bar{y}; \rho, \tau; tt') = \frac{4h^2}{(1+h)^2(1-tt'a^2)^2} \exp i \frac{\rho((1+a^2)y+2a\bar{y})-2a^2\rho\tau}{1-tt'a^2} , \quad (4.56)$$

where

$$a_{\alpha\dot{\alpha}} = \frac{\lambda x_{\alpha\dot{\alpha}}}{1 + \sqrt{1 - \lambda^2 x^2}} . \quad (4.57)$$

We can thus write

$$K_\mu = \frac{1}{4i} Q \omega_\mu^{\alpha\beta} \left[(1+a^2)^2 y_\alpha y_\beta + 4(1+a^2) a_\alpha^{\dot{\alpha}} y_\beta \bar{y}_{\dot{\alpha}} + 4 a_\alpha^{\dot{\alpha}} a_\beta^{\dot{\beta}} \bar{y}_{\dot{\alpha}} \bar{y}_{\dot{\beta}} \right] - \text{h.c.} , \quad (4.58)$$

where

$$Q = -\frac{1}{4}(1-a^2)^2 \int_{-1}^1 dt \int_{-1}^1 dt' \frac{q(t)q(t')(1+t)(1+t')}{(1-tt'a^2)^4} . \quad (4.59)$$

This function is studied further and evaluated at order ν^2 in Appendix D. From (4.52) and (4.58) it follows that all higher-spin gauge fields vanish,

$$W_\mu = 0 , \quad (4.60)$$

while the vierbein and Lorentz connection are given by

$$\omega_\mu^{\alpha\beta} = f \omega_{\mu(0)}^{\alpha\beta} , \quad (4.61)$$

$$e_\mu^{\alpha\dot{\alpha}} = e_{\mu(0)}^{\alpha\dot{\alpha}} - (Qf + \bar{Q}\bar{f}) \left[4a^2 e_{\mu(0)}^{\alpha\dot{\alpha}} + \frac{(1+a^2)^4}{(1-a^2)^2} \lambda^3 dx^a x_a x^{\alpha\dot{\alpha}} \right] , \quad (4.62)$$

where we have used $a_\alpha{}^\dot{\alpha} a_\beta{}^\dot{\beta} \bar{\omega}_{\dot{\alpha}\dot{\beta}}^{(0)} = a^2 \omega_{\alpha\beta}^{(0)}$, and defined

$$f = \frac{1 + (1 - a^2)^2 \bar{Q}}{|1 + (1 + a^2)^2 Q|^2 - 16a^4 |Q|^2}. \quad (4.63)$$

We identify the vielbein as a conformally rescaled AdS_4 metric,

$$e^a = f_1 dx^a + \lambda^2 f_2 dx^b x_b x^a = \frac{2\Omega d(g_1 x^a)}{1 - \lambda^2 g_1^2 x^2}, \quad (4.64)$$

where

$$f_1(x^2) = \frac{2(1 + a^2)^2}{(1 - a^2)^2} (1 - 4a^2(Qf + \bar{Q}\bar{f})) , \quad f_2(x^2) = \frac{(1 + a^2)^4}{(1 - a^2)^2} (Qf + \bar{Q}\bar{f}) , \quad (4.65)$$

and the scale factor is given by

$$\Omega = \frac{(1 - \lambda^2 g_1^2 x^2) f_1}{2g_1} , \quad g_1 = \exp \left[-\lambda^2 \int_{x^2}^{\lambda^{-2}} \frac{f_2(t) dt}{f_1(t)} \right]. \quad (4.66)$$

In the boundary region, where $a^2 \rightarrow 1$, the double integral in (4.59) diverges at $t = t' = \pm 1$ while the pre-factor goes to zero, as to produce a finite residue given by

$$\lim_{a^2 \rightarrow 1} Q = -\frac{\nu^2}{6}. \quad (4.67)$$

In this limit

$$\lim_{a^2 \rightarrow 1} \Omega = \frac{1}{1 - \frac{4\nu^2}{3}}. \quad (4.68)$$

This factor is positive in the Type B model, while curiously enough it blows up at a critical value, within the range (D.6), in the Type A model.

By the Z_2 -symmetry, the metric in the \tilde{x}^a -coordinate chart is given by the same functions as in the x^a -coordinate system (which is not the same as a reparameterization!). In the distant past and future, where $a^2 \rightarrow -1$, the function Q diverges logarithmically leading to qualitatively different behavior of the scale factors in the cases of the Type A and B models, which is analyzed in more detail in [25]. Here we content ourselves by observing that any pathological behavior in the strong-coupling region $\lambda^2 x^2 < -1$ can be removed by going to the dual weakly coupled frame. Thus, geometrically speaking, the solution interpolates between two asymptotically AdS_4 regions at $\lambda^2 x^2 \sim 1$ and $\lambda^2 \tilde{x}^2 \sim 1$ via an interior given by complicated scale factors (discussed in [25]) times foliates determined by symmetries, to which we shall turn our attention next.

4.4 SYMMETRIES OF THE SOLUTION

A more general discussion of symmetries of solutions will be given in Section 5. In view of (5.4), the gauge transformations preserving the primed solution obey

$$\widehat{D}'_\alpha \tilde{\epsilon}' = 0, \quad \pi(\tilde{\epsilon}') = \tilde{\epsilon}', \quad (4.69)$$

where the last condition is equivalent to $[\tilde{\epsilon}', \widehat{\Phi}']_\pi = 0$ since $\widehat{\Phi}' = \nu$ is constant. The condition (4.69) is by construction solved by the full $SO(3, 1)$ generators, *i.e.*

$$\tilde{\epsilon}' = \frac{1}{4i} \Lambda^{\alpha\beta} \widehat{M}'_{\alpha\beta} - \text{h.c.}, \quad (4.70)$$

with $\widehat{M}'_{\alpha\beta}$ given by (2.24) and constant $\Lambda_{\alpha\beta}$.

The solution is also left invariant by additional transformations with rigid higher-spin parameters

$$\tilde{\epsilon}' = \sum_{\ell=0}^{\infty} \tilde{\epsilon}'_\ell, \quad (4.71)$$

where the ℓ 'th level is given by

$$\tilde{\epsilon}'_\ell = \sum_{m+n=2\ell+1} \Lambda^{\alpha_1 \dots \alpha_{2m}, \dot{\alpha}_1 \dots \dot{\alpha}_{2n}} \widehat{M}'_{\alpha_1 \alpha_2} \star \dots \star \widehat{M}'_{\alpha_{2m-1} \alpha_{2m}} \star \widehat{M}'_{\dot{\alpha}_1 \dot{\alpha}_2} \star \dots \star \widehat{M}'_{\dot{\alpha}_{2n-1} \dot{\alpha}_{2n}} - \text{h.c.}, \quad (4.72)$$

with constant $\Lambda^{\alpha_1 \dots \alpha_{2m}, \dot{\alpha}_1 \dots \dot{\alpha}_{2n}}$. These parameters span the solution space to (4.69), provided that this space has a smooth dependence on ν . The full symmetry algebra is thus a higher-spin extension of $SO(3, 1) \simeq SL(2, C)$, that we shall denote by

$$hsl(2, C; \nu) \supset sl(2, C), \quad (4.73)$$

where $sl(2, C)$ is generated by $\widehat{M}'_{\alpha\beta}$ and its hermitian conjugate, and we have indicated that in general the structure coefficients may depend on the deformation parameter ν .

The generators of the $SO(3, 1)$ transformations preserving the space-time dependent field configuration, that is, obeying (5.1), are by construction given by

$$\widehat{M}^L_{\alpha\beta} = L^{-1}(x) \star \widehat{M}'_{\alpha\beta} \star L(x), \quad (4.74)$$

and are related to the full Lorentz generators $\widehat{M}_{\alpha\beta}$ by

$$\widehat{M}^L_{\alpha\beta} - \widehat{M}_{\alpha\beta} = \widetilde{M}_{\alpha\beta}(\lambda x) - M_{\alpha\beta}, \quad (4.75)$$

where $M_{\alpha\beta} = y_\alpha y_\beta$ and we have defined

$$\widetilde{M}_{\alpha\beta}(v) = L^{-1}(\lambda^{-1}v) \star M_{\alpha\beta} \star L(\lambda^{-1}v) = \tilde{y}_\alpha(v) \tilde{y}_\beta(v), \quad (4.76)$$

where the transformed oscillators are defined by

$$\tilde{y}_\alpha(v) = \frac{y_\alpha + v_\alpha \dot{\alpha} \bar{y}_\alpha}{\sqrt{1 - v^2}} , \quad (4.77)$$

and obey the same algebra as the original oscillators, *viz.*

$$\tilde{y}_\alpha \star \tilde{y}_\beta = \tilde{y}_\alpha \tilde{y}_\beta + i\epsilon_{\alpha\beta} . \quad (4.78)$$

If we let \hat{g}'_Λ and \hat{g}^L_Λ be the group elements generated by $\hat{M}'_{\alpha\beta}$ and $\hat{M}^L_{\alpha\beta}$, respectively, then it follows that

$$L(x) \star \hat{g}_\Lambda = \hat{g}'_\Lambda \star L(\Lambda x) , \quad (\Lambda x^\mu) = \Lambda^\mu_\nu x^\nu . \quad (4.79)$$

The spacetime decomposes under this $SO(3, 1)$ action into orbits that are three-dimensional hyper-surfaces which describe local foliations of AdS_4 with dS_3 and H_3 spaces in the regions $x^2 > 0$ and $x^2 < 0$, respectively.

5 ON OTHER SOLUTIONS WITH NON-MAXIMAL ISOMETRY

In this section we discuss the consequences of imposing various non-maximal symmetry conditions on solutions. We first do this in a general setting, and then consider symmetry groups of dimensions 3, 4 and 6. In this approach, we recover the previous $so(3, 1)$ -invariant solution and also construct new solutions at the first order in the Weyl zero-form. The latter include domain walls and rotationally invariant solutions.

5.1 SOME GENERALITIES

The solution with maximal unbroken symmetry is the AdS_4 vacuum $\hat{\Phi} = 0$, which is invariant under rigid $hs(4)$ transformations, with Z -independent parameters. Let us consider non-vanishing $\hat{\Phi}$ that is invariant under a non-trivial set of transformations with parameters belonging to

$$h(\hat{\Phi}) = \left\{ \hat{\epsilon} : \quad \hat{D}\hat{\epsilon} = 0 , \quad [\hat{\epsilon}, \hat{\Phi}]_\pi = 0 , \quad \tau(\hat{\epsilon}) = \hat{\epsilon}^\dagger = -\hat{\epsilon} \right\} . \quad (5.1)$$

As is the case for $hs(4)$, this algebra closes under \star -commutation of parameters as well as compositions induced by the associativity of the \star -product, *e.g.*

$$\hat{\epsilon}_1 \star \hat{\epsilon}_2 \star \hat{\epsilon}_3 + \hat{\epsilon}_3 \star \hat{\epsilon}_2 \star \hat{\epsilon}_1 , \quad \hat{\epsilon}_i \in h(\hat{\Phi}) . \quad (5.2)$$

In the case that $h(\widehat{\Phi})$ contains a finite-dimensional rank- r subalgebra $\mathbf{g}_r \subset SO(3, 2)$ this induces a natural higher-spin structure $h(\widehat{\Phi})$, as exemplified in (4.71) and (4.72) for $hsl(2, C; \nu)$.

In the Z -space approach

$$\widehat{\epsilon} = \widehat{L}^{-1} \star \widehat{\epsilon}' \star \widehat{L} , \quad \partial_\mu \widehat{\epsilon}' = 0 , \quad (5.3)$$

where

$$\widehat{D}'_\alpha \widehat{\epsilon}' = 0 , \quad [\widehat{\epsilon}', \widehat{\Phi}']_\pi = 0 . \quad (5.4)$$

Expanding perturbatively in $\widehat{\Phi}'|_{Z=0} = C'$,

$$\widehat{\epsilon}' = \epsilon' + \widehat{\epsilon}'_{(1)} + \widehat{\epsilon}'_{(2)} + \cdots , \quad (5.5)$$

and assuming that the parameters obeying (5.4) are given up to and including order $n-1$ ($n = 1, 2, \dots$), then $\widehat{\epsilon}'_{(n)}$ is determined by

$$(\widehat{D}'_\alpha \widehat{\epsilon}')_{(n)} = 0 , \quad (5.6)$$

which is an integrable partial differential equation in Z -space provided that

$$\widehat{I}'_{(n)} \equiv ([\widehat{\epsilon}', \widehat{\Phi}']_\pi)_{(n)} = 0 . \quad (5.7)$$

Using the Z -space field equations obeyed in the lower orders, one can show that

$$\partial_\alpha \widehat{I}'_{(n)} = 0 , \quad (5.8)$$

so that (5.7) holds if ($n = 1, 2, \dots$)

$$I'_{(n)} \equiv \widehat{I}'_{(n)}|_{Z=0} = \sum_{p+q=n} [\widehat{\epsilon}'_{(p)}, \widehat{\Phi}'_{(q)}]_\pi|_{Z=0} = 0 . \quad (5.9)$$

In the first order, the symmetry condition reads

$$[\epsilon', C']_\pi = 0 . \quad (5.10)$$

We shall denote the stability algebra of C'

$$h(C') = \left\{ \epsilon' : [\epsilon', C']_\pi = 0 , \quad \tau(\epsilon') = (\epsilon')^\dagger = -\epsilon' \right\} . \quad (5.11)$$

As found in the case of $hsl(2, C; \nu)$, the full symmetry algebra $h(\widehat{\Phi})$ is in general a deformed version of $h(C')$. Moreover, we shall denote the space of all twisted-adjoint elements invariant under $h(C')$ by $B(C')$, *i.e.*

$$B(C') = \left\{ \tilde{C}' : [\epsilon', \tilde{C}']_\pi = 0 , \quad \forall \epsilon' \in h(C') \right\} . \quad (5.12)$$

Covariance implies that if g is an $hs(4)$ group element, then

$$h(g^{-1} \star C' \star \pi(g)) = g^{-1} \star h(C') \star g. \quad (5.13)$$

The spaces $B(g^{-1} \star C' \star \pi(g))$ and $g^{-1} \star B(C') \star \pi(g)$ are in general not isomorphic, however, as there exist special points in the twisted-adjoint representation space where $\dim B(C')$ is less than the generic value on the group orbit. The simplest example is the point $C' = \nu$, as will be discussed below (5.51). Another subtlety, that we shall exemplify below, is related to the fact that the associativity of the \star -product implies that if C' is a regular initial condition then $B(C')$ contains the elements $C'_{(2n+1)}$ defined by (3.32). Thus, if $\dim B(C')$ is finite then C' must either violate regularity at some level of perturbation theory or be projector-like in the sense that $C'_{2n+1} \sim C'$ for some finite value of n .

In the case of a solution in which $\hat{\Phi}$ asymptotes to $L^{-1} \star C' \star L$ with $h(C') \supset \mathbf{g}_r \subset SO(3, 2)$, where (3.28) is the AdS_4 gauge function (3.28), the solution has \mathbf{g}_r isometry close to the boundary provided the perturbation theory holds. Since the space-time field equations are manifestly diffeomorphism and locally Lorentz invariant, the \mathbf{g}_r -isometry extends to the solution in the interior, where it acts on the full master fields via parameters $\hat{\epsilon} = L^{-1} \star \hat{\epsilon}' \star L$. Hence, the integrability conditions (5.9) must hold, resulting in a full symmetry algebra $h(\hat{\Phi})$ that is in general some deformed higher-spin extension of \mathbf{g}_r with deformation parameters given by C' .

Let us next examine the above features in more detail in some special cases.

5.2 SOLUTIONS WITH UNBROKEN $SO(3, 1)$ SYMMETRY

Acting on the exact $SO(3, 1)$ -invariant solution described in Section 4 with the gauge transformation generated by the group element

$$g(v) = L(\lambda^{-1}v), \quad (5.14)$$

with L given by (4.1), which requires

$$v^2 < 1, \quad (5.15)$$

one finds the gauge-equivalent exact solution given by the zero-form

$$\hat{\Phi}'^{(v)} = g^{-1}(v) \star \hat{\Phi}' \star \pi(g(v)) = \frac{\nu}{b_1} (1 - v^2) \exp(iyv\bar{y}), \quad (5.16)$$

and the internal connection

$$\hat{A}'^{(v)}_\alpha = \frac{i}{2} \int_{-1}^1 dt [q^{(+)}(t) - q^{(-)}(t)] g^{-1}(v) \star (z_\alpha e^{\frac{i}{2}(1+t)u}) \star g(v). \quad (5.17)$$

The gauge-transformed solution has

$$C'^{(v)} = \widehat{\Phi}'^{(v)}, \quad (5.18)$$

with stability group $h(C'^{(v)})$ generated via enveloping of the generators $\widetilde{M}_{\alpha\beta}(v)$ given by (4.76). Thus, the space $B(C'^{(v)})$ consists of all elements obeying

$$[\tilde{y}_\alpha(v)\tilde{y}_\beta(v), \tilde{C}']_\pi = 0, \quad (5.19)$$

amounting to the following second order partial differential equation

$$\frac{4}{1-v^2} \left(iy_\alpha \partial_\beta + i(v\bar{y})_\alpha (v\bar{\partial})_\beta + y_\alpha (v\bar{y})_\beta - \partial_\alpha (v\bar{\partial})_\beta \right) \tilde{C}' + (\alpha \leftrightarrow \beta) = 0. \quad (5.20)$$

This equation can be solved using the ansatz

$$\tilde{C}' = \tilde{C}'(V), \quad V = yv\bar{y}, \quad (5.21)$$

implying the following second-order ordinary differential equation in variable V with constant coefficients

$$\left(-v^2 \frac{d^2}{dV^2} + i(1+v^2) \frac{d}{dV} + 1 \right) \tilde{C}' = 0, \quad (5.22)$$

which admits the solutions

$$\tilde{C}' = \begin{cases} \tilde{\nu}_1 e^{iV} + \tilde{\nu}_2 e^{i\frac{V}{v^2}} & \text{for } v^2 \neq 0 \text{ and } v^2 < 1 \\ \tilde{\nu}_1 e^{iV} & \text{for } v^2 = 0 \end{cases}. \quad (5.23)$$

It is not possible to produce any further solutions to (5.19) using (3.32), since $\exp iV \star \pi(\exp iV)$ is proportional to 1, while the \star -product composition of $\exp iV$ and $\pi(\exp i\frac{V}{v^2})$ is divergent. Thus,

$$\dim B(C'^{(v)}) = \begin{cases} 2 & \text{for } v^2 \neq 0 \text{ and } v^2 < 1 \\ 1 & \text{for } v^2 = 0 \end{cases}. \quad (5.24)$$

We stress that $h(C'^{(v)})$ is the stability group of the twisted-adjoint element $C'^{(v)}$, and not of the parameter v , and that therefore the stability group is still $SO(3, 1)$ when v is null. The additional solution with super-luminal boost-parameter v^a is not gauge-equivalent to $\widehat{\Phi}'^{(v)}$, and that \tilde{C}' is regular or singular as a initial condition, according to the terminology introduced in Section 3.3, depending on whether $\tilde{\nu}_1\tilde{\nu}_2 = 0$ or $\tilde{\nu}_1\tilde{\nu}_2 \neq 0$, respectively. Whether the singular or super-luminal cases can be elevated to exact solutions remains to be seen.

For $v^2 \neq 0$ the space $B(C'(v))$ is invariant under $v^a \leftrightarrow v^a/v^2$. Extending (5.23) to

$$v^2 > 1 , \quad (5.25)$$

gives an $SO(3, 1)$ -invariant two-dimensional solution space with stability group generated by $\tilde{M}_{\alpha\beta}(v)$ given by (4.76), where the fact that the denominator in (4.77) is imaginary does not present an obstacle since the $sl(2, C)$ -doublet oscillators are complex.

The “self-dual” case $v^2 = 1$ requires a separate treatment. Here $\eta_\alpha = y_\alpha + v_\alpha{}^{\dot{\alpha}}\bar{y}_{\dot{\alpha}}$ is a commuting oscillator giving rise to a three-dimensional translation generator $p_a = (v^b(\sigma_{ab})^{\alpha\beta}\eta_\alpha\eta_\beta + \text{h.c.})$ obeying $[p_a, p_b]_\star = 0$ and $v^a p_a = 0$. The commuting oscillators obey the reality condition $(\eta_\alpha)^\dagger = v_{\dot{\alpha}}{}^\alpha\eta_\alpha$, and transform as doublets under the oscillator realization of the $SL(2, R) \simeq SO(2, 1)$ that leaves v_a invariant. This leads to an $ISO(2, 1)$ -invariant two-dimensional solution space $B(e^V)$ to be described next at the level of the linearized field equations (see eq. (5.47)) together with some other interesting reductions.

5.3 ON DOMAIN WALLS, ROTATIONALLY INVARIANT AND RW-LIKE SOLUTIONS

Here we shall discuss some classes of solutions of considerable interest corresponding to the rank- r subalgebras $\mathbf{g}_r \subset SO(3, 2)$ parameterized by

$$\mathbf{g}_3 : M_{ij} = L_i^a L_j^b M_{ab} , \quad (5.26)$$

$$\mathbf{g}_4 : M_{ij} , \quad P = L^a P_a = \frac{1}{4} L^a (\sigma_a)^{\alpha\dot{\alpha}} y_\alpha \bar{y}_{\dot{\alpha}} , \quad (5.27)$$

$$\mathbf{g}_6 : M_{ij} , \quad P_i = (\alpha M_{ab} L^b + \beta P_a) L_i^a , \quad (5.28)$$

where α and β are real parameters and, in all cases, (L_i^a, L^a) is a representative of the coset $SO(3, 1)/SO(3)$ or $SO(3, 1)/SO(2, 1)$, obeying

$$L^a L_a = \epsilon = \pm 1 , \quad L_i^a L_a = 0 , \quad L_i^a L_{ja} = \eta_{ij} = \text{diag}(+, +, -\epsilon) . \quad (5.29)$$

The $SO(3, 2)$ algebra (A.2) yields $[M, M] \sim M$. Furthermore, for \mathbf{g}_4 one finds $[M, P] \sim 0$, while for \mathbf{g}_6 one finds $[M, P] \sim P$, and

$$[P_i, P_j] = i(\beta^2 - \epsilon\alpha^2)M_{ij} . \quad (5.30)$$

In summary, one has

$$\mathbf{g}_6 = \begin{cases} SO(3, 1) & \text{for } \alpha^2 - \epsilon\beta^2 > 0 , \quad \epsilon = \pm 1 \\ ISO(2, 1) & \text{for } \beta^2 - \alpha^2 = 0 , \quad \epsilon = +1 \\ SO(2, 2) & \text{for } \beta^2 - \alpha^2 > 0 , \quad \epsilon = +1 \end{cases} \quad (5.31)$$

$$\mathbf{g}_4 = \begin{cases} SO(3) \times SO(2) , & \epsilon = -1 \\ SO(2,1) \times SO(2) , & \epsilon = +1 \end{cases} \quad (5.32)$$

$$\mathbf{g}_3 = \begin{cases} SO(3) , & \epsilon = -1 \\ SO(2,1) , & \epsilon = +1 \end{cases} \quad (5.33)$$

Next we seek \mathbf{g}_r -invariant twisted-adjoint initial conditions $C'(y, \bar{y}) = \hat{\Phi}'|_{Z=0}$ that obey the invariance condition (5.9) in *the first order, viz.*

$$[\epsilon', C']_\pi = 0 , \quad \epsilon' \in \mathbf{g}_r . \quad (5.34)$$

As discussed in Section 3.2, C' provides an initial condition to (3.16)-(3.18) giving rise to a full master field $\hat{\Phi}'$ related to the space-time Weyl tensors through $\hat{\Phi} = L^{-1} \star \hat{\Phi}' \star L^{-1}$, so that $C = L^{-1} \star C' \star L^{-1}$ are the linearized Weyl tensors, discussed in Section 3.1. We note that C is invariant under gauge transformations with parameters $L^{-1} \star \epsilon' \star L$, while $\hat{\Phi}$ is invariant under gauge transformations with deformed parameters $\hat{\epsilon}'$ provided that the integrability conditions (5.9) hold to all orders. In Section 5.4 we shall perform the symmetry analysis in the second order.

The condition (5.34) decomposes into two irreducible conditions

$$[M_{ij}, C']_\star = 0 , \quad \begin{cases} [P_i, C']_\pi = 0 & \text{for } \mathbf{g}_6 \\ \{P, C'\}_\star = 0 & \text{for } \mathbf{g}_4 \end{cases} . \quad (5.35)$$

The first condition can be shown to have the general solution

$$\mathbf{g}_3 : \quad C' = C'(P) , \quad (5.36)$$

where $C'(P)$ is a function that we shall assume is analytic at the origin. To arrive at this conclusion one can use the fact that the oscillator realization (A.3) implies $M_{[ab} M_{c]d} = M_{[ab} P_{c]} = M_{ab} M_{c}^b = M_{ab} P^b = 0$, or, alternatively, note that the only \mathbf{g}_3 -invariant spinorial objects are $\epsilon_{\alpha\beta}$, $\epsilon_{\dot{\alpha}\dot{\beta}}$ and $L_{\alpha\dot{\alpha}}$.

Thus, the \mathbf{g}_3 -invariant solution space $B(C')$, defined by (5.12), is infinite-dimensional, and indeed closes under (3.32) for regular initial data. The \mathbf{g}_3 -invariance can be imposed at the full level using the deformed generators

$$\mathbf{g}_3 : \quad \widehat{M}'_{ij} = L_i^a L_j^b \widehat{M}'_{ab} , \quad (5.37)$$

where \widehat{M}'_{ab} are obtained from (2.24). This results in consistent $SO(3)$ -invariant or $SO(2,1)$ -invariant “mini-superspace” truncations described by the master equations (3.16)-(3.18) with reduced master fields

$$\hat{\Phi}' = \hat{\Phi}'(P; u, \bar{u}; P', \Pi, \bar{\Pi}) , \quad (5.38)$$

and

$$\widehat{A}'_\alpha = z_\alpha \widehat{A}_1 + y_\alpha \widehat{A}_2 + L_\alpha{}^{\dot{\alpha}} (\bar{z}_{\dot{\alpha}} \widehat{A}_3 + \bar{y}_{\dot{\alpha}} \widehat{A}_4) , \quad \widehat{A}_i = \widehat{A}_i(P; u, \bar{u}; P', \Pi, \bar{\Pi}) , \quad (5.39)$$

taken to depend on the oscillators only through the following set of composite variables,

$$u = y^\alpha z_\alpha , \quad P = \frac{1}{4} L^{\alpha\dot{\alpha}} y_\alpha \bar{y}_{\dot{\alpha}} , \quad P' = \frac{1}{4} L^{\alpha\dot{\alpha}} z_\alpha \bar{z}_{\dot{\alpha}} , \quad \Pi = L^a (\sigma_a)^{\alpha\dot{\alpha}} z_\alpha \bar{y}_{\dot{\alpha}} , \quad (5.40)$$

where $L_{\alpha\dot{\alpha}} = (\sigma^a)_{\alpha\dot{\alpha}} L_a$. We stress that the full \mathbf{g}_3 symmetry is manifest, assuring that the above restricted dependence on the oscillators is actually consistent with the master equations (3.16)-(3.18). The reduced models correspond geometrically to local reductions on either a two-sphere or a two-hyperboloid, with complete symmetry algebra given by deformed enveloping-algebra extensions of \mathbf{g}_3 generated by \widehat{M}'_{ij} .

Turning to the second set of conditions in (5.35), we use the lemmas

$$L_i{}^a L^p [M_{ab}, P^k]_\star = ik\epsilon L_i{}^a P_a P^{k-1} , \quad (5.41)$$

$$L_i{}^a \{P_a, P^k\}_\star = L_i{}^a P_a \left(2P^k - \frac{k(k-1)\epsilon}{8} P^{k-2} \right) , \quad (5.42)$$

$$\{P, P^k\}_\star = 2P^{k+1} + \frac{k(k+1)\epsilon}{8} P^{k-1} , \quad (5.43)$$

to rewrite them as the following second-order differential equations

$$\mathbf{g}_6 : \left(-\frac{\beta\epsilon}{8} \frac{d^2}{dP^2} + i\alpha\epsilon \frac{d}{dP} + 2\beta \right) C'(P) = 0 , \quad (5.44)$$

$$\mathbf{g}_4 : \left(\frac{\epsilon}{8} \frac{d^2}{dP^2} + 2 \right) PC'(P) = 0 . \quad (5.45)$$

The resulting initial data that are analytic at $P = 0$ are given by

$$SO(3, 1) : C'(P) = \nu_1 e^{4i\rho_- P} + \nu_2 e^{4i\rho_+ P} , \quad (5.46)$$

$$ISO(2, 1) : C'(P) = (\mu_1 + i\mu_2 P) e^{4iP} , \quad (5.47)$$

$$SO(2, 2) : C'(P) = (\nu_1 + i\nu_2) e^{4i\rho_- P} + (\nu_1 - i\nu_2) e^{4i\rho_+ P} , \quad (5.48)$$

$$SO(3) \times SO(2) : C'(P) = \nu_1 \frac{\sinh(4P)}{P} , \quad (5.49)$$

$$SO(2, 1) \times SO(2) : C'(P) = \nu_1 \frac{\sin(4P)}{P} , \quad (5.50)$$

where $\nu_{1,2}$ and $\mu_{1,2}$ are real constants and

$$\rho_{\pm} = \frac{\alpha}{\beta} \pm \sqrt{\left(\frac{\alpha}{\beta}\right)^2 - \epsilon} . \quad (5.51)$$

We see that in general $\dim B(C') = 2$ for the \mathbf{g}_6 -invariant initial data, except at a few special points where $\dim B(C') = 1$, while $\dim B(C') = 1$ for the \mathbf{g}_4 -invariant initial data.

In the case of $SO(3, 1)$, we can identify particular cases of (5.46) with the exact solutions (5.16) and (5.17), namely⁶

$$\begin{aligned} \epsilon = 1, \quad 1 > v^2 > 0 : \quad & \begin{cases} L^a = \frac{v^a}{\sqrt{v^2}}, \quad \rho_+ = -\sqrt{v^2}, \quad \nu_2 = \frac{\nu}{b_1}(1 - v^2), \quad \nu_1 = 0 \\ L^a = -\frac{v^a}{\sqrt{v^2}}, \quad \rho_- = \sqrt{v^2}, \quad \nu_1 = \frac{\nu}{b_1}(1 - v^2), \quad \nu_2 = 0 \end{cases}, \\ \epsilon = -1, \quad v^2 < 0 : \quad & \begin{cases} L^a = -\frac{v^a}{\sqrt{-v^2}}, \quad \rho_+ = \sqrt{-v^2}, \quad \nu_2 = \frac{\nu}{b_1}(1 - v^2), \quad \nu_1 = 0 \\ L^a = \frac{v^a}{\sqrt{-v^2}}, \quad \rho_- = -\sqrt{-v^2}, \quad \nu_1 = \frac{\nu}{b_1}(1 - v^2), \quad \nu_2 = 0 \end{cases}. \end{aligned}$$

On the other hand, as noted in Section 5.2, the initial condition $C' = \tilde{\nu} e^{iyv\bar{y}}$ with $v^2 > 1$ and $\tilde{\nu}$ an arbitrary constant, and the initial conditions with $\nu_1\nu_2 \neq 0$ are gauge-inequivalent to the above exact solutions. In these cases one might expect the full zero-form $\hat{\Phi}'$ to receive Z -dependent higher-order corrections. In the perturbative approach, the case of $\nu_1\nu_2 \neq 0$ is singular in the sense defined in Section 3.3, while the case of $v^2 > 1$ involves poles at $t = 1/v^2 < 1$ that can be avoided using suitable contours in the t -plane.

Turning to the $ISO(2, 1)$ -invariant case (5.47), we identify it as a singular initial condition for a flat domain wall solution. The case with $\mu_2 = 0$ can be obtained formally as the limit of the dS_3 domain wall in which

$$v^2 \rightarrow 1, \quad \left| \frac{\nu}{b_1} \right| \rightarrow \infty, \quad \mu_1 = \frac{\nu(1 - v^2)}{b_1} \text{ fixed}. \quad (5.52)$$

It remains to be seen whether the internal connection (5.17) and the space-time gauge fields are well-behaved in this limit. In view of (D.6), we expect there to be differences between the limiting procedures in the Type A and the Type B model, where ν is real and imaginary, respectively.

The remaining three types of initial conditions listed above, *i.e.* the those invariant under $SO(2, 2)$, $SO(3) \times SO(2)$ and $SO(2, 1) \times SO(2)$, are singular for all values of the parameters.

⁶The case of $v^2 = 0$ requires the embedding of $SO(3, 1)$ into $SO(3, 2)$ using (4.76).

In the singular cases found above, the perturbative expansion can be obtained using the closed contours given in (3.34) and (3.35). It remains to be seen, however, whether full solutions can be given explicitly in these cases. Another issue to settle is whether the ϵ' -parameters can be deformed into full $\tilde{\epsilon}'$ -parameters obeying (5.4), to which we turn our attention next.

5.4 EXISTENCE OF SYMMETRY PARAMETERS TO THE SECOND ORDER

Let us prove the existence of the second correction $\tilde{\epsilon}'_{(2)}$ to the $\tilde{\epsilon}'$ -parameter associated to the \mathbf{g}_r -symmetries discussed above. To do so, we need to establish the integrability of (5.4) to second order, *i.e.* verify (5.9) for $n = 2$, where the relevant quantity reads

$$I'_{(2)} \equiv \left([\tilde{\epsilon}'_{(1)}, C']_\pi + [\epsilon', \hat{\Phi}'_{(2)}]_\pi \right) \Big|_{Z=0} , \quad (5.53)$$

with $\epsilon' \in \mathbf{g}_r$ given in (5.28) or (5.27), and C' in (5.46)-(5.50).

In fact, it is possible to show that (5.53) vanishes for the more general case with parameter

$$\epsilon' = \eta' - \eta'^\dagger , \quad \eta' = \frac{1}{4i} (\lambda^{\alpha\beta} y_\alpha y_\beta + \lambda^{\alpha\dot{\alpha}} y_\alpha \bar{y}_{\dot{\alpha}}) , \quad (5.54)$$

and twisted-adjoint element

$$C' = \sum_{m=0}^{\infty} i^m f_m v_{\alpha(m)\dot{\alpha}(m)} y^{\alpha(m)} \bar{y}^{\dot{\alpha}(m)} , \quad (5.55)$$

where f_m are real numbers; we use the notation $y^{\alpha(m)} = y^{\alpha_1} \cdots y^{\alpha_m}$; and we have defined

$$v_{\alpha(m)\dot{\alpha}(m)} = v^{a_1} \cdots v^{a_m} (\sigma_{a_1})_{\alpha_1\dot{\alpha}_1} \cdots (\sigma_{a_m})_{\alpha_m\dot{\alpha}_m} . \quad (5.56)$$

The first-order correction to the parameter is given by

$$\hat{\epsilon}_{(1)} = \hat{\eta}_{(1)} - \hat{\eta}_{(1)}^\dagger , \quad (5.57)$$

with

$$\hat{\eta}_{(1)} = -b_1 \int_0^1 dt \int_0^1 t' dt' \left(\frac{itt'}{2} \lambda^{\alpha\beta} z_\alpha z_\beta + \lambda^{\alpha\dot{\beta}} z_\alpha \bar{\partial}_{\dot{\beta}} \right) C'(-tt' z, \bar{y}) e^{itt' y z} , \quad (5.58)$$

and the second-order correction to the zero-form is given in (C.5), which can be re-written as

$$\hat{\Phi}'_{(2)} = T \hat{B} , \quad (5.59)$$

where the twisted-adjoint projection map T is defined by

$$T\hat{f} = \hat{f} + \tau\bar{\pi}\hat{f} + \pi(\hat{f} + \tau\bar{\pi}\hat{f})^\dagger, \quad (5.60)$$

and

$$\hat{B} = -z^\alpha \int_0^1 dt (\hat{A}_\alpha^{(1)} \star C')_{Z \rightarrow tZ}, \quad (5.61)$$

with $\hat{A}_\alpha^{(1)}$ given by (C.3). The quantity $I'_{(2)}$, which is a twisted-adjoint element, can thus be written as

$$I'_{(2)} = T(\hat{B}_1 + \hat{B}_2)|_{Z=0}, \quad (5.62)$$

where

$$\hat{B}_1 = \hat{\eta}'_{(1)} \star C', \quad (5.63)$$

and

$$\hat{B}_2 = T[\epsilon', \hat{B}]_\pi. \quad (5.64)$$

The expansion of \hat{B}_1 reads

$$\begin{aligned} \hat{B}_1 = & -b_1 \int_0^1 dt \int_0^1 t' dt' \sum_{m,n} (-tt')^m i^{m+n} f_m f_n v^{\alpha(m)\dot{\alpha}(m)} v^{\beta(n)\dot{\beta}(n)} \times \\ & \times \left(\left(\frac{itt'}{2} \lambda^{\gamma\delta} z_\gamma z_\delta + \lambda^{\gamma\dot{\delta}} z_\gamma \bar{\partial}_{\dot{\delta}} \right) z_{\alpha(m)} \bar{y}_{\dot{\alpha}(m)} e^{itt'yz} \right) \star y_{\beta(n)} \bar{y}_{\dot{\beta}(n)}. \end{aligned} \quad (5.65)$$

To calculate the $\lambda^{\gamma\delta}$ -contributions to $T\hat{B}_1|_{Z=0}$ we let $n \rightarrow m+2+k$ and contract $m+2$ of the y_β -oscillators with the z -oscillators in the first factor. The remaining y_β -oscillators may contract z -oscillators in the exponent, so we sum over l such contractions with $0 \leq l \leq k$. The resulting terms have the common l -independent structure

$$\begin{aligned} & T \left(\lambda^{\gamma\delta} v^{\alpha(m)\dot{\alpha}(m)} v_{\alpha(m)\beta(k)\gamma\delta\dot{\beta}(m+2+k)} y^{\beta(k)} (\bar{y}_{\dot{\alpha}(m)} \star \bar{y}^{\dot{\beta}(m+2+k)}) \right) \\ & \sim T \left(\lambda^{\gamma\delta} v_{\gamma\delta\alpha(k)\dot{\alpha}(k+2)} y^{\alpha(k)} \bar{y}^{\dot{\alpha}(k+2)} \right) = 0, \end{aligned} \quad (5.66)$$

where we use (5.56) and the fact that $y^{\alpha(k)} \bar{y}^{\dot{\alpha}(k+2)}$ cannot contribute to the twisted-adjoint representation. Similarly, to calculate the $\lambda^{\gamma\dot{\delta}}$ -contributions we let $n \rightarrow m+1+k$, resulting in

$$\begin{aligned} & T \left(\lambda^{\gamma\dot{\alpha}} v^{\alpha(m)\dot{\alpha}(m)} v_{\gamma\alpha(m)\beta(k)\dot{\alpha}(m+1+k)} y^{\beta(k)} (\bar{y}_{\dot{\alpha}(m-1)} \star \bar{y}^{\dot{\beta}(m+1+k)}) \right) \\ & \sim T \left(\lambda^{\gamma\dot{\alpha}} v_{\gamma\beta(k)\dot{\beta}(k+1)} y^{\beta(k)} \bar{y}^{\dot{\alpha}\dot{\beta}(k+1)} \right) = 0. \end{aligned} \quad (5.67)$$

The same type of cancellations occur in $T\hat{B}_2|_{Z=0}$. Here the $\lambda^{\gamma\delta}$ -contribution to $[\epsilon', \hat{B}]_\pi$ is given by

$$\begin{aligned} & \frac{b_1}{8} \int_0^1 dt \int_0^1 t' dt' \sum_{m,n} (-t')^m i^{m+n} f_m f_n v^{\alpha(m)\dot{\alpha}(m)} v^{\beta(n)\dot{\beta}(n)} \times \\ & \times \lambda^{\gamma\delta} \left[y_\gamma y_\delta, z^\alpha \left((z_{\alpha(m+1)} \bar{y}_{\dot{\alpha}(m)} e^{it'yz}) \star (y_{\beta(n)} \bar{y}_{\dot{\beta}(n)}) \right)_{Z \rightarrow tZ} \right]_*, \end{aligned} \quad (5.68)$$

which we evaluate at $Z = 0$ using

$$\lambda^{\gamma\delta}[y_\gamma y_\delta, z^\alpha \hat{f}]_\star|_{Z=0} = 4\lambda^{\gamma\alpha} \partial_\gamma^{(y)} \hat{f}|_{Z=0} , \quad (5.69)$$

resulting in contributions to $T[\epsilon', \hat{B}]_\pi|_{Z=0}$ of the form

$$\begin{aligned} & T \left(\lambda^{\alpha\beta} v^{\alpha(m)\dot{\alpha}(m)} v_{\alpha(m+1)\beta(k)\dot{\beta}(m+1+k)} y^{\beta(k-1)} (\bar{y}_{\dot{\alpha}(m)} \star \bar{y}^{\dot{\beta}(m+1+k)}) \right) \\ & \sim T \left(\lambda^{\alpha\beta} v_{\alpha\beta(k)\dot{\beta}(k+1)} y^{\beta(k-1)} \bar{y}^{\dot{\beta}(k+1)} \right) = 0 . \end{aligned} \quad (5.70)$$

Finally, the $\lambda^{\gamma\dot{\delta}}$ -contribution to $[\epsilon', \hat{B}]_\pi$ reads

$$\begin{aligned} & \frac{b_1}{4} \int_0^1 dt \int_0^1 t' dt' \sum_{m,n} (-t')^m i^{m+n} f_m f_n v^{\alpha(m)\dot{\alpha}(m)} v^{\beta(n)\dot{\beta}(n)} \times \\ & \times \lambda^{\gamma\dot{\delta}} \left\{ y_\gamma \bar{y}_\dot{\delta}, z^\alpha \left((z_{\alpha(m+1)} \bar{y}_{\dot{\alpha}(m)} e^{it'yz}) \star (y_{\beta(n)} \bar{y}_{\dot{\beta}(n)}) \right)_{Z \rightarrow tZ} \right\}_\star , \end{aligned} \quad (5.71)$$

which we evaluate at $Z = 0$ using

$$\lambda^{\gamma\dot{\delta}} \{y_\gamma \bar{y}_\dot{\delta}, z^\alpha \hat{f}\}_\star|_{Z=0} = -2i\lambda^{\alpha\dot{\delta}} \bar{y}_\dot{\delta} \hat{f} , \quad (5.72)$$

resulting in contributions to $T[\epsilon', \hat{B}]_\pi|_{Z=0}$ of the form

$$\begin{aligned} & T \left(\lambda^{\alpha\dot{\delta}} \bar{y}_\dot{\delta} v^{\alpha(m)\dot{\alpha}(m)} v_{\alpha(m+1)\beta(k)\dot{\beta}(m+1+k)} y^{\beta(k)} (\bar{y}_{\dot{\alpha}(m)} \star \bar{y}^{\dot{\beta}(m+1+k)}) \right) \\ & \sim T \left(\lambda^{\alpha\dot{\delta}} \bar{y}_\dot{\delta} v_{\alpha\beta(k)\dot{\beta}(k+1)} y^{\beta(k)} \bar{y}^{\dot{\beta}(k+1)} \right) = 0 , \end{aligned} \quad (5.73)$$

which concludes the proof of (5.53).

Interestingly enough, we have found a stronger result, namely that

$$C' = C'(V) , \quad V = yv\bar{y} , \quad (5.74)$$

preserves $SO(3, 2)$ to second order in the curvature expansion for general initial conditions $C'(V)$. We expect $SO(3, 2)$ to be broken down to some \mathbf{g}_r with $r \leq 6$ at third order.

6 DISCUSSION

We have seen that, while the field equations written in traditional form in spacetime are highly complicated, and indeed not even available explicitly at present, we can nonetheless solve them exactly by exploiting their relative simple form in terms of master fields. Doing so, we have found the first exact solution of higher spin gauge theory in $D > 3$ other than the anti de Sitter spacetime.

The solution presented here forms a consistent background for the first-quantized topological open twistor string description of higher spin gauge theory [17]. It would be interesting to study this 1+1 dimensional field theory which provides a framework for a dual twistor space interpretation of the solution.

It would also be useful to determine the holographic interpretation of our solution. This requires, however, the knowledge of an on-shell action and a systematic way to label the fluctuation fields that takes into account the infinite dimensional nature of the underlying symmetries. While progress has been made in that direction [27] much remains to be done to develop fully the necessary tools to facilitate the fluctuation analysis.

Next, we note that while our solution is time dependent, its cosmological interpretation is not straightforward. To begin with, an understanding of the key concepts of horizons and singularities are very much based on the geodesic equation of motion for test particles, yet we do not know its higher spin covariant counterpart.

A key feature of the theory relevant in dealing with both holographic and spacetime geometric aspects is the presence of nonlocalities in the spacetime field equations. These nonlocalities should be manageable, however, once we work with master fields. For example, using this approach, we have already found certain zero-form charges that provide a set of labels for classical solutions. The master field approach should also be used to construct a higher spin dressed version of a spacetime line element, which in turn should be embedded into a manifestly higher spin covariant infinite dimensional geometry.

Finally, it is interesting to compare our solution with the $SO(3, 1)$ invariant solution to the gauged $\mathcal{N} = 8, D = 4$ supergravity found in [28]. To this end, we first observe that the minimal bosonic models we have studied here are consistent truncations of the higher-spin gauge theory based on $shs(8|4) \supset osp(8|4)$ [29] which contains, respectively, $35_+ + 35_-$ scalars and pseudo-scalars in the supergravity multiplet and $1 + \bar{1}$ scalar and pseudo-scalar in an $s_{\max} = 4$ multiplet that we refer to as the Konishi multiplet. The truncations to the Type A and Type B models keep the Konishi scalar and pseudo-scalar, respectively [30]. Thus we see that an important difference between the solution of [28] and ours is that different scalars have been activated. The full consequences of this difference remains to be investigated.

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A CONVENTIONS AND NOTATION

We use the conventions of [29] in which the $SO(3, 2)$ generators obey

$$[M_{AB}, M_{CD}] = i\eta_{BC}M_{AD} + 3 \text{ more} , \quad M_{AB} = (M_{AB})^\dagger , \quad (\text{A.1})$$

with $\eta_{AB} = \text{diag}(- - + + +)$. The commutation relations decompose into

$$[M_{ab}, M_{cd}]_\star = 4i\eta_{[c|[b}M_{a]d]} , \quad [M_{ab}, P_c]_\star = 2i\eta_{c[b}P_{a]} , \quad [P_a, P_b]_\star = iM_{ab} . \quad (\text{A.2})$$

The oscillator realization is taken to be

$$M_{ab} = -\frac{1}{8} \left[\sigma_{ab}^{\alpha\beta} y_\alpha y_\beta + (\bar{\sigma}_{ab})^{\dot{\alpha}\dot{\beta}} \bar{y}_{\dot{\alpha}} \bar{y}_{\dot{\beta}} \right] , \quad P_a = \frac{1}{4} (\sigma_a)^{\alpha\dot{\beta}} y_\alpha \bar{y}_{\dot{\beta}} , \quad (\text{A.3})$$

where the van der Warden symbols obey

$$\begin{aligned} (\sigma^a)_\alpha^{\dot{\alpha}} (\bar{\sigma}^b)_{\dot{\alpha}}^\beta &= \eta^{ab} \delta_\alpha^\beta + (\sigma^{ab})_\alpha^\beta , & (\bar{\sigma}^a)_{\dot{\alpha}}^\alpha (\sigma^b)_\alpha^{\dot{\beta}} &= \eta^{ab} \delta_{\dot{\alpha}}^{\dot{\beta}} + (\bar{\sigma}^{ab})_{\dot{\alpha}}^{\dot{\beta}} , \\ \frac{1}{2} \epsilon_{abcd} (\sigma^{cd})_{\alpha\beta} &= i(\sigma_{ab})_{\alpha\beta} , & \frac{1}{2} \epsilon_{abcd} (\bar{\sigma}^{cd})_{\dot{\alpha}\dot{\beta}} &= -i(\bar{\sigma}_{ab})_{\dot{\alpha}\dot{\beta}} , \end{aligned} \quad (\text{A.4})$$

$$((\sigma^a)_{\alpha\dot{\beta}})^\dagger = (\bar{\sigma}^a)_{\dot{\alpha}\alpha} = (\sigma^a)_{\alpha\dot{\alpha}} , \quad ((\sigma^{ab})_{\alpha\beta})^\dagger = (\bar{\sigma}^{ab})_{\dot{\alpha}\dot{\beta}} , \quad (\text{A.5})$$

with Minkowski space-time metric $\eta_{ab} = \text{diag}(- + + +)$, and spinor conventions $A^\alpha = \epsilon^{\alpha\beta} A_\beta$, $A_\alpha = A^\beta \epsilon_{\beta\alpha}$, and $(A^\alpha)^\dagger = \bar{A}^{\dot{\alpha}}$.

The $SO(3, 2)$ -valued connection is expressed as

$$\Omega = \frac{1}{4i} dx^\mu \left[\omega_\mu^{\alpha\beta} y_\alpha y_\beta + \bar{\omega}_\mu^{\dot{\alpha}\dot{\beta}} \bar{y}_{\dot{\alpha}} \bar{y}_{\dot{\beta}} + 2e_\mu^{\alpha\dot{\beta}} y_\alpha \bar{y}_{\dot{\beta}} \right] . \quad (\text{A.6})$$

The components of $\mathcal{R} = d\Omega + \Omega \wedge \star\Omega$ are

$$\mathcal{R}_{\alpha\beta} = d\omega_{\alpha\beta} + \omega_{\alpha\gamma} \wedge \omega_\beta^\gamma + e_{\alpha\dot{\delta}} \wedge e_\beta^{\dot{\delta}} , \quad (\text{A.7})$$

$$\mathcal{R}_{\dot{\alpha}\dot{\beta}} = d\bar{\omega}_{\dot{\alpha}\dot{\beta}} + \bar{\omega}_{\dot{\alpha}\dot{\gamma}} \wedge \bar{\omega}_{\dot{\beta}}^{\dot{\gamma}} + e_{\delta\dot{\alpha}} \wedge e_{\dot{\beta}}^\delta , \quad (\text{A.8})$$

$$\mathcal{R}_{\alpha\dot{\beta}} = de_{\alpha\dot{\beta}} + \omega_{\alpha\gamma} \wedge e_\beta^{\gamma\dot{\beta}} + \bar{\omega}_{\dot{\beta}\dot{\delta}} \wedge e_\alpha^{\alpha\dot{\delta}} . \quad (\text{A.9})$$

Defining

$$\omega^{\alpha\beta} = -\frac{1}{4} (\sigma_{ab})^{\alpha\beta} \omega^{ab} , \quad \omega^{\dot{\alpha}\dot{\beta}} = -\frac{1}{4} (\bar{\sigma}_{ab})^{\dot{\alpha}\dot{\beta}} \omega^{ab} , \quad e^{\alpha\dot{\alpha}} = -\frac{\lambda}{2} (\sigma_a)^{\alpha\dot{\alpha}} e^a , \quad (\text{A.10})$$

where λ is the inverse radius of the AdS_4 vacuum, and converting the spinor indices of the curvatures in the same way, gives

$$\mathcal{R}^{ab} = d\omega^{ab} + \omega^a_c \omega^{cb} + \lambda^2 e^a \wedge e^b , \quad \mathcal{R}^a = de^a + \omega^a_b e^b . \quad (\text{A.11})$$

It follows that the AdS_4 vacuum $\Omega_{(0)}$ is characterized by

$$R_{(0)\mu\nu,\rho\sigma} = -\lambda^2 \left(g_{(0)\mu\rho} \bar{g}_{(0)\nu\sigma} - \bar{g}_{(0)\nu\rho} \bar{g}_{(0)\mu\sigma} \right) , \quad R_{(0)\mu\nu} = -3\lambda^2 g_{(0)\mu\nu} . \quad (\text{A.12})$$

B GAUSSIAN INTEGRATION FORMULAE

To compose coset representatives we make use of the formula

$$e^{iy a \bar{y}} \star e^{iy b \bar{y}} = \frac{1}{\det(1 + \bar{a} \bar{b})} \exp i \left[y \left(a \frac{1}{1 + \bar{b} a} + b \frac{1}{1 + \bar{a} b} \right) \bar{y} - y \frac{a \bar{b}}{1 + \bar{a} \bar{b}} y + \bar{y} \frac{\bar{b} a}{1 + \bar{b} a} \bar{y} \right], \quad (\text{B.1})$$

where we use matrix notation, *e.g.* $ya\bar{y} = y^\alpha a_\alpha{}^\beta \bar{y}_\beta$ and $y\bar{a}by = y^\alpha \bar{a}_\alpha{}^\beta b_\beta{}^\gamma y_\gamma$, with $a_{\alpha\beta} = a^\mu(\sigma_\mu)_{\alpha\beta}$ and $\bar{a}_{\dot{\alpha}\dot{\beta}} = a^\mu(\sigma_\mu)_{\dot{\alpha}\dot{\beta}}$. In deriving this formula, we use (2.5) and perform the integrals over holomorphic and anti-holomorphic variables *independently*. In doing so, the exponentials remain separately linear in integration variables, allowing the use of the identity

$$\int \frac{d^2\xi d^2\eta}{(2\pi)^2} e^{i\eta^\alpha [\xi_\alpha + u_\alpha(\bar{\xi}, \bar{\eta})] + i\xi^\alpha v_\alpha(\bar{\xi}, \bar{\eta})} = e^{iu^\alpha(\bar{\xi}, \bar{\eta})v_\alpha(\bar{\xi}, \bar{\eta})}, \quad (\text{B.2})$$

and its analog for anti-holomorphic variables. Other useful formulae are

$$e^{iy a \bar{y}} \star e^{i\rho y} \star e^{iy a \bar{y}} = \frac{1}{(1 + a^2)^2} \exp i \left[\frac{2ya\bar{y} + (1 - a^2)\rho y}{1 + a^2} \right], \quad (\text{B.3})$$

$$(e^{-itz a \bar{y}} e^{ity z}) \star e^{iy a \bar{y}} = \frac{1}{(1 - ta^2)^2} \exp i \left[\frac{(1 - a^2)tyz + (1 - t)ya\bar{y}}{1 - ta^2} \right]. \quad (\text{B.4})$$

The relation (4.15) follows from the lemma

$$\begin{aligned} & e^{i[t y z + \rho y + \tau z]} \star e^{i[t' y z + \rho' y + \tau' z]} \\ &= e^{i[(t + t' - 2t')yz + ((1 - t')\rho + (1 - t)\rho' + t\tau' - t'\tau)y + ((1 - t')\tau + (1 - t)\tau' + t\rho' - t'\rho)z + (\rho + \tau)(\rho' - \tau')]} \end{aligned} \quad (\text{B.5})$$

Finally, to evaluate K_μ on the solution, we need

$$\begin{aligned} & e^{iy a \bar{y}} \star e^{i(s y z + \rho y + \tau z)} \star e^{-iy a \bar{y}} \Big|_{Z=0} \\ &= \frac{1}{(1 - (1 - 2s)a^2)^2} \exp i \left[\frac{\rho((1 + a^2)y + 2a\bar{y}) - 2a^2\rho\tau}{1 - (1 - 2s)a^2} \right], \end{aligned} \quad (\text{B.6})$$

where $a^2 = a^\mu a_\mu$.

C SECOND-ORDER CORRECTIONS TO THE FIELD EQUATIONS

The second-order source terms $\hat{P}_\mu^{(2)}$ and $\hat{J}_{\mu\nu}^{(2)}$ in the field equations are given by [6]

$$P_\mu^{(2)} = \Phi \star \bar{\pi}(W_\mu) - W_\mu \star \Phi$$

$$+ \left. \left(\Phi \star \bar{\pi}(\hat{e}_\mu^{(1)}) - \hat{e}_\mu^{(1)} \star \Phi + \hat{\Phi}^{(2)} \star \bar{\pi}(e_\mu) - e_\mu \star \hat{\Phi}^{(2)} \right) \right|_{Z=0} , \quad (\text{C.1})$$

$$\begin{aligned} J_{\mu\nu}^{(2)} &= - \left[\left. \left(\hat{e}_\mu^{(1)} \star \hat{e}_\nu^{(1)} + \{e_\mu, \hat{e}_\nu^{(2)}\}_\star + \{e_\mu, \hat{W}_\nu^{(1)}\}_\star + \{W_\mu, \hat{e}_\nu^{(1)}\}_\star \right) \right|_{Z=0} \right. \\ &\quad \left. + \left. \left(i R_{\mu\nu}^{\alpha\beta} \hat{A}_\alpha^{(1)} \star \hat{A}_\beta^{(1)} + \text{h.c.} \right) \right|_{Z=0} + W_\mu \star W_\nu \right] - (\mu \leftrightarrow \nu) , \quad (\text{C.2}) \end{aligned}$$

where the hatted quantities are defined as⁷

$$\hat{A}_\alpha^{(1)} = -\frac{ib_1}{2} z_\alpha \int_0^1 t dt \Phi(-tz, \bar{y}) \kappa(tz, y) , \quad (\text{C.3})$$

$$\begin{aligned} \hat{A}_\alpha^{(2)} &= z_\alpha \int_0^1 t dt \left(\hat{A}^{(1)\beta} \star \hat{A}_\beta^{(1)} \right)_{Z \rightarrow tZ} + \bar{z}^{\dot{\beta}} \int_0^1 t dt \left[\hat{A}_\alpha^{(1)}, \hat{A}_{\dot{\beta}}^{(1)} \right]_{\star Z \rightarrow tZ} \\ &\quad - \frac{ib_1}{2} \int_0^1 t dt \left(\hat{\Phi}^{(2)} \star \kappa \right)_{Z \rightarrow tZ} , \quad (\text{C.4}) \end{aligned}$$

$$\hat{\Phi}^{(2)} = \int_0^1 dt \left(z^\alpha \left(\Phi \star \bar{\pi}(\hat{A}_\alpha^{(1)}) - \hat{A}_\alpha^{(1)} \star \Phi \right)_{Z \rightarrow tZ} + \bar{z}^{\dot{\alpha}} \left(\Phi \star \pi(\hat{A}_{\dot{\alpha}}^{(1)}) - \hat{A}_{\dot{\alpha}}^{(1)} \star \Phi \right)_{Z \rightarrow tZ} \right) , \quad (\text{C.5})$$

$$\hat{e}_\mu^{(1)} = -ie_\mu^{\alpha\dot{\alpha}} \int_0^1 \frac{dt}{t} \left(\left[\bar{y}_{\dot{\alpha}}, \hat{A}_\alpha^{(1)} \right]_* + \left[\hat{A}_{\dot{\alpha}}^{(1)}, y_\alpha \right]_* \right)_{Z \rightarrow tZ} , \quad (\text{C.6})$$

$$\begin{aligned} \hat{W}_\mu^{(1)} &= -i \int_0^1 \frac{dt}{t} \left(\left[\frac{\partial W_\mu}{\partial y^\alpha}, \hat{A}^{\alpha(1)} \right]_* + \left[\hat{A}^{\dot{\alpha}(1)}, \frac{\partial W_\mu}{\partial \bar{y}^{\dot{\alpha}}} \right]_* \right)_{Z \rightarrow tZ} , \quad (\text{C.7}) \\ &\quad (\text{C.8}) \end{aligned}$$

$$\begin{aligned} \hat{e}_\mu^{(2)} &= -ie_\mu^{\alpha\dot{\alpha}} \int_0^1 \frac{dt}{t} \left(\left[\bar{y}_{\dot{\alpha}}, \hat{A}_\alpha^{(2)} \right]_* + \left[\hat{A}_{\dot{\alpha}}^{(2)}, y_\alpha \right]_* \right)_{Z \rightarrow tZ} \\ &\quad - e_\mu^{\alpha\dot{\alpha}} \int_0^1 \frac{dt}{t} \int_0^1 \frac{dt'}{t'} \left(\hat{A}^{\beta(1)} \star \left(\frac{\partial}{\partial z^\beta} - \frac{\partial}{\partial \bar{y}^\beta} \right) \left(\left[\bar{y}_{\dot{\alpha}}, \hat{A}_\alpha^{(1)} \right]_* + \left[\hat{A}_{\dot{\alpha}}^{(1)}, y_\alpha \right]_* \right)_{Z \rightarrow t'Z} \right. \\ &\quad \left. + \hat{A}^{\dot{\beta}(1)} \star \left(\frac{\partial}{\partial z^{\dot{\beta}}} + \frac{\partial}{\partial \bar{y}^{\dot{\beta}}} \right) \left(\left[\bar{y}_{\dot{\alpha}}, \hat{A}_\alpha^{(1)} \right]_* + \left[\hat{A}_{\dot{\alpha}}^{(1)}, y_\alpha \right]_* \right)_{Z \rightarrow t'Z} \right. \\ &\quad \left. + \left(\frac{\partial}{\partial z^\beta} + \frac{\partial}{\partial \bar{y}^\beta} \right) \left(\left[\bar{y}_{\dot{\alpha}}, \hat{A}_\alpha^{(1)} \right]_* + \left[\hat{A}_{\dot{\alpha}}^{(1)}, y_\alpha \right]_* \right)_{Z \rightarrow t'Z} \star \hat{A}^{\beta(1)} \right) \end{aligned} \quad (\text{C.9})$$

⁷In eq. (C.4) the last term was inadvertently omitted in [6].

$$+ \left(\frac{\partial}{\partial \bar{z}^{\dot{\beta}}} - \frac{\partial}{\partial \bar{y}^{\dot{\beta}}} \right) \left([\bar{y}_{\dot{\alpha}}, \hat{A}_{\alpha}^{(1)}]_* + [\hat{A}_{\dot{\alpha}}^{(1)}, y_{\alpha}]_* \right)_{Z \rightarrow t'Z} \star \hat{A}^{\dot{\beta}(1)} \Big)_{Z \rightarrow tZ} ,$$

where the replacement $(z, \bar{z}) \rightarrow (tz, t\bar{z})$ in a quantity must be performed *after* the quantity has been written in Weyl ordered form i.e. after the \star products defining the quantity has been performed.

D ANALYSIS OF THE Q -FUNCTION

Expanding the denominator of (4.59), and splitting into even and odd parts in a^2 , as

$$Q = Q_+ + Q_- , \quad Q_{\pm}(-a^2) = \pm Q_{\pm}(a^2) , \quad (\text{D.1})$$

we find

$$Q_+ = -(1-a^2)^2 \sum_{p=0}^{\infty} \binom{-4}{2p} a^{4p} \left[\int_0^1 dt t^{2p} (q^{(+)}(t) - tq^{(-)}(t)) \right]^2 , \quad (\text{D.2})$$

$$Q_- = (1-a^2)^2 \sum_{p=0}^{\infty} \binom{-4}{2p+1} a^{4p+2} \left[\int_0^1 dt t^{2p+1} (tq^{(+)}(t) - q^{(-)}(t)) \right]^2 , \quad (\text{D.3})$$

with $q^{(\pm)}(t)$ given by (4.38) and (4.43). Performing the integrals we arrive at

$$Q_+ = -\frac{(1-a^2)^2}{4} \sum_{p=0}^{\infty} \binom{-4}{2p} a^{4p} \left(\sqrt{1 - \frac{\nu}{2p+1}} - \sqrt{1 + \frac{\nu}{2p+3}} \right)^2 \quad (\text{D.4})$$

$$Q_- = \frac{(1-a^2)^2}{4} \sum_{p=0}^{\infty} \binom{-4}{2p+1} a^{4p+2} \left(\sqrt{1 - \frac{\nu}{2p+3}} - \sqrt{1 + \frac{\nu}{2p+3}} \right)^2 , \quad (\text{D.5})$$

that are valid for

$$-3 \leq \text{Re } \nu \leq 1 . \quad (\text{D.6})$$

Expanding in ν ,

$$Q_+ = \sum_{n=2}^{\infty} \nu^n Q_{+,n} , \quad Q_- = \sum_{n=2}^{\infty} \nu^{2n} Q_{+,2n} , \quad (\text{D.7})$$

the first non-trivial contribution is found to be

$$Q_{+,2} = -\frac{(1-a^2)^2}{8} \left(\frac{d^2}{da^4} a^4 \right)^2 \left[{}_3F_2(1, 1, 4; 3, 3; -a^2) + {}_3F_2(1, 1, 4; 3, 3; a^2) \right] \quad (\text{D.8})$$

$$Q_{-,2} = \frac{(1-a^2)^2}{8} \left[{}_3F_2(2, 2, 4; 3, 3; -a^2) - {}_3F_2(2, 2, 4; 3, 3; a^2) \right] . \quad (\text{D.9})$$

We note that parity acts in tangent space by exchanging holomorphic and anti-holomorphic oscillators and, moreover, by exchanging $\hat{\Phi}$ with $\hat{\Phi}$ or $-\hat{\Phi}$ in the Type A and Type B models, respectively. Since the spin $s = 2, 4, \dots$ Weyl tensors $C_{\alpha_1 \dots \alpha_{2s}}$ can be taken to be even under parity, this means that the Type B model is invariant under $\phi \rightarrow -\phi$, or, equivalently, under $\nu \rightarrow -\nu$, while this need not be, and is indeed not, a symmetry in the Type A model.

References

- [1] M.A. Vasiliev, “Consistent equations for interacting gauge fields of all spins in 3 + 1 dimensions”, *Phys. Lett.* **243** (1990) 378.
- [2] M. A. Vasiliev, “Higher spin gauge theories: star-product and AdS space”, [hep-th/9910096](#).
- [3] M.A. Vasiliev, “ Nonlinear equations for symmetric massless higher spin fields in $(A)dS_d$ ”, *Phys. Lett. B* **567** (2003) 139, [hep-th/0304049](#).
- [4] A. Sagnotti, E. Sezgin and P. Sundell, “On higher spins with a strong $Sp(2,R)$ condition”, [hep-th/0501156](#).
- [5] X. Bekaert, S. Cnockaert, C. Iazeolla and M. A. Vasiliev, “Nonlinear higher spin theories in various dimensions”, [hep-th/0503128](#).
- [6] E. Sezgin and P. Sundell, “Analysis of higher spin field equations in four dimensions”, *JHEP* **0207** (2002) 055, [hep-th/0205132](#).
- [7] E. Sezgin and P. Sundell, “Holography in 4D (super) higher spin theories and a test via cubic scalar couplings”, *JHEP* **0507** (2005) 044, [hep-th/0305040](#).
- [8] F. Kristiansson and P. Rajan, “Scalar field corrections to $AdS(4)$ gravity from higher spin gauge theory,” *JHEP* **0304** (2003) 009, [hep-th/0303202](#).
- [9] M. A. Vasiliev, “More on equations of motion for interacting massless fields of all spins in 3+1 dimensions”, *Phys. Lett. B* **285** (1992) 225.
- [10] E. Sezgin and P. Sundell, “Doubletons and 5D higher spin gauge theory”, *JHEP* **0109** (2001) 036, [hep-th/0105001](#).
- [11] E. Sezgin and P. Sundell, “Towards massless higher spin extension of $D=5$, $N=8$ gauged supergravity”, *JHEP* **0109** (2001) 025, [hep-th/0107186](#).
- [12] E. Sezgin and P. Sundell, “7D bosonic higher spin theory: symmetry algebra and linearized constraints”, *Nucl. Phys. B* **644** (2002) 303 [Erratum-ibid. B **660** (2003) 403], [hep-th/0112100](#).
- [13] M. A. Vasiliev, “Higher spin superalgebras in any dimension and their representations,” *JHEP* **0412** (2004) 046, [hep-th/0404124](#).
- [14] M. Cederwall, “AdS twistors for higher spin theory,” *AIP Conf. Proc.* **767** (2005) 96, [hep-th/0412222](#).

- [15] I. Bars and M. Picon, “Single twistor description of massless, massive, AdS, and other interacting particles,” [hep-th/0512091](#).
- [16] I. Bars and S. J. Rey, “Noncommutative $Sp(2,R)$ gauge theories: A field theory approach to two-time physics,” *Phys. Rev. D* **64** (2001) 046005, [hep-th/0104135](#).
- [17] J. Engquist and P. Sundell, “Brane partons and singleton strings”, [hep-th/0508124](#).
- [18] B. Sundborg, “Stringy gravity, interacting tensionless massless higher-spins”, *Nucl. Phys. Proc. Suppl.* **102** (2001) 113, [hep-th/0103247](#).
- [19] E. Sezgin and P. Sundell, “Massless higher spins and holography”, *Nucl. Phys. B* **644** (2003) 303; Erratum-*ibid. B* **660** (2003) 403, [hep-th/0205131](#).
- [20] I.R. Klebanov and A.M. Polyakov, “AdS dual of the critical $O(N)$ vector model”, *Phys. Lett. B* **550** (2002) 213, [hep-th/0210114](#).
- [21] A. Sagnotti and M. Tsulaia, “On higher spins and the tensionless limit of string theory”, *Nucl. Phys. B* **682** (2004) 83, [hep-th/0311257](#).
- [22] N. Beisert, M. Bianchi, J. F. Morales and H. Samtleben, “Higher spin symmetry and $N = 4$ SYM”, *JHEP* **0407** (2004) 058, [hep-th/0405057](#).
- [23] S. F. Prokushkin and M. A. Vasiliev, “Higher-spin gauge interactions for massive matter fields in 3D AdS space-time”, *Nucl. Phys. B* **545** (1999) 385, [hep-th/9806236](#).
- [24] K. I. Bolotin and M. A. Vasiliev, “Star-product and massless free field dynamics in $AdS(4)$ ”, *Phys. Lett. B* **479** (2000) 421, [hep-th/0001031](#).
- [25] E. Sezgin and P. Sundell, “On an Exact Cosmological Solution of Higher Spin Gauge Theory”, [hep-th/](#).
- [26] M. A. Vasiliev, “Extended higher spin superalgebras and their realizations in terms of quantum operators”, *Fortsch. Phys.* **36** (1988) 33.
- [27] M. A. Vasiliev, “Actions, charges and off-shell fields in the unfolded dynamics approach,” [hep-th/0504090](#).
- [28] T. Hertog and G. T. Horowitz, “Towards a big crunch dual,” *JHEP* **0407** (2004) 073, [hep-th/0406134](#); “Holographic description of AdS cosmologies,” *JHEP* **0504**, (2005) 005, [hep-th/0503071](#).

- [29] E. Sezgin and P. Sundell, “Higher spin $N = 8$ supergravity”, JHEP **bf 9811** (1998) 016, [hep-th/9805125](#).
- [30] J. Engquist, E. Sezgin and P. Sundell, “On $N = 1,2,4$ higher spin gauge theories in four dimensions,” Class. Quant. Grav. **19** (2002) 6175, [hep-th/0207101](#).